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FINAL REPORT ON CONTRACT NASw-688

Study of
Abort Procedures, Lunar Landing
and
Lunar Reconnaissance
for the Apollo Mission

C60-10
6 September 1963

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C60-10

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GENERAL INTRODUCTION

This final report on NASA Study Contract No. NASw-688 is a compilation of the analyses and investigations conducted at Bissett-Berman on Guidance and Navigation for Apollo.

Early in the contract a procedure of submitting technical notes to NASA Headquarters immediately upon completion of each phase of analysis or investigation was established in order to make the results of the study rapidly available to NASA. This report is essentially a collection of these notes, organized into four main categories:

I. Ground Support to Apollo in Aborts

II. Far-Side Relay

III. CM/SM Abort Guidance

IV. Lunar Landing

Some of the notes previously submitted to NASA are not included in this report since they were produced primarily for learning purposes or for internal information.

The majority of the effort has been devoted to investigating the capability of the MSFN for orbit determination and therefore, the preponderance of notes deal with this task. The feasibility of using the S-IVB booster to provide a far-side relay is the subject of several notes included in Section II. The problems of CM/SM abort guidance and lunar landing are treated in Sections III and IV, respectively. Since Section IV, Lunar Landing is classified, it is contained under separate cover.

As recommended by NASA Headquarters personnel, a double precision program for calculating the covariance matrices of the maximum likelihood estimators of the orbit parameters is being prepared, and when the results of the computer runs are available, a summary report will be prepared. The summary report will contain data on the capability of the MSFN for orbit determination under the following sets of conditions:

a. Range-rate data only from one station (no a priori information).

b. Range data only from one station (no a priori information).

c. Range and range-rate data from one station (no a priori information).

d. Combination of range and range-rate from multiple stations (no a priori information).

e. The effect of a priori information on a, b, c, and d.

f. The effect of intermediate boosts.

Some preliminary results of the computer analyses are presented at the end of Part I of this report. It is expected that the completed summary report of the computer error analyses will be available in about 2 to 3 months.

PART I

GROUND SUPPORT TO APOLLO IN ABORTS

PART I: GROUND SUPPORT TO APOLLO IN ABORTS

INTRODUCTION TO PART I

Ground system support to the Apollo vehicles in an abort situation is strongly dependent upon the ability of the ground system to determine the orbits of the Apollo vehicles.

A study was initiated to discover how well the ground support system can determine the Apollo vehicle orbits about the Moon. At the start of this study, the orbit determinations were to be made using range-rate data from a single DSIF station.

The DSIF range-rate capabilities were determined[1/], and a number of rough analyses performed to find out if, indeed, these DSIF measurements would be useful. These rough analyses considered simplified situations to make solutions to the problems tractable. They included several analyses for determining the in-plane orbit parameters assuming the orbit plane orientation known, or determining all the orbit parameters except rotation of the orbit plane about the line-of-sight.[2/]

1/ Bissett-Berman Corporation Apollo Note No. 17, DSIF Accuracy and Report C60-6, DSIF Capability for Apollo Guidance and Navigation (C).

2/ Bissett-Berman Corporation Apollo Notes No.

32. Derivation of Four LEM Orbit Parameters from Doppler Data Only.
40. Use of the DSIF to Determine Orbits About the Lunar Surface.
41. Generalized Future Range Linear Error Coefficient for the Restricted Two-Body Problem.
50. A General Technique to Derive Smoothing Error Coefficient for the Restricted Two-Body Problem.
60. Error Analysis for Determining In-Plane Orbit Parameters Using DSIF Doppler Measurements During Descent Into Synchronous Orbit.
61. Determination of Selenocentric Orbit Parameters with DSIF.
64. Capability of the DSIF for Determining In-Plane Orbit Parameters During Ascent.
67. Calculation of Covariance Matrices, I.
73. Some Additional Error Calculations for Determining Three In-Plane Orbit Parameters from DSIF Doppler Measurements.
80. Correction to Calculation of Covariance Matrices, I., and Report C60-6, DSIF Capability for Apollo Guidance and Navigation.

Two other analyses were performed to determine the rotation of the orbit plane about the line-of-sight, assuming the other parameters were known.[3/]

These analyses indicated that it was likely that the DSIF could be of assistance in determination of orbits about the Moon, so a more comprehensive analysis to determine all six orbit parameters was initiated. This was done for range-rate data from a single observing station, and included motion of the station about the Earth and motion of the Moon about the Earth. These motions must be included to make determination of the orbit orientation possible under these conditions. The general method of analysis is indicated in Bissett-Berman Corporation Apollo Note No. 43, The Calculation of the Covariance Matrix of the Maximum Likelihood Estimators of Orbit Parameters Obtained from Range-Rate Data*. The details of the analysis leading to a computer program to perform the necessary computations are given in Apollo Note No. 82, Calculation of Covariance Matrices III.* The analysis of Apollo Note No. 82 is of special interest in that the problem has been formulated in such a fashion that results are still computable when the orbit eccentricity approaches or equals zero.

About this time there was a redirection of effort on this contract, with emphasis on ground support system assistance to the LEM during aborts, and characterized by relatively short total times of observation so that observing station motion and motion of the Moon in its orbit were no longer important considerations. The analysis including these motions, however, is still employed.

3/ Bissett-Berman Corporation Apollo Notes No.

37. Accuracy of Measuring CM/SM Position in Lunar Orbit Relative to a Lunar Landmark by Optical Sighting.

48. CM Orbit Orientation, and Report

C60-6, DSIF Capability for Apollo Guidance and Navigation.

* Titles marked with an asterisk are Apollo Notes included in this Final Report.

At the same time, it was suggested that range and angle data from observing stations also be employed, and that observations from multiple stations be employed simultaneously. The necessary analysis appears in Apollo Notes No. 77, Calculation of Covariance Matrices for Multiple Uncorrelated Data Sources*, No. 83, Use of Range and Range-Rate Data,* and No. 94, Use of Angle Data.

The question of whether the choice of a time origin affects the standard deviations of estimated quantities has arisen several times in the course of this control. Apollo Note No. 85, The Equivalence of Data Processing Schemes in Linearized Error Analysis * shows that the choice of time origin has no effect.

As part of the problem of assisting the LEM in ascent, it is necessary to be able to make use of a priori information or information telemetered from the Apollo vehicles. The means for orbit parameter covariance matrix determination on the basis of all this information is spelled out in Apollo Note No. 95, Ground Assistance to LEM, Including Mid-Course Correction* which indicates how to make use of a priori information and telemetered boost data, and Apollo Note No. 96, Use of LEM/CM Observations* which indicates how the usefulness of radar or visual sightings from the CM/SM can be included by a very slight modification of the technique employed to make use of observations from Earth.

Apollo Note No. 99, Preliminary Results of Computer Analyses*, includes the results of computations to date.

The question of whether the ground support system can process its observations fast enough to be of assistance in LEM aborts has arisen several times in the course of this contract. The question is answered in Apollo Note No. 93, Ground System Computation of LEM Orbits* in which it is shown that the actual delay due to computation is small.

Before it was decided to restrict ascent orbit considerations to the (present) nominal Hohmann transfer, two analyses were performed in a search for an optimal abort trajectory. One of these [4/] used a modified Hohmann transfer followed by additional orbitings of the Moon before rendezvous. The other [5/] chose an ascent scheme that would allow for much larger guidance errors, trading increased time to rendezvous for savings in fuel in the case of large guidance errors.

For awhile it seemed as though rendezvous and docking might be the most critical portion of the abort, so aids to rendezvous in case of equipment failures were discussed.[6/] These included use of the ground stations in docking, ring-a-round techniques for measurement of scalar range and range-rate between vehicles, use of the LEM landing radar, and an optical technique for obtaining range-rate between vehicles.

4/ Bissett-Berman Corporation Apollo Note No.

79. An Approach to Estimating the Allowable Injection Errors for the DSIF Aided Rendezvous Scheme.

5/ 81. Maximum Allowable Injection Errors for a Particular DSIF Aided Rendezvous Scheme.

6/ 87. Rendezvous Aids.

APOLLO NOTE NO. 43

J. Holdsworth
15 April 1963

THE CALCULATION OF THE COVARIANCE MATRIX OF THE MAXIMUM LIKELIHOOD ESTIMATORS OF ORBIT PARAMETERS OBTAINED FROM RANGE RATE DATA

The purpose of this note is to enable one to determine the accuracy with which a set of M orbit parameters may be determined from observations made upon some measurable quantity such as range, range rate, etc. More precisely, it is assumed that at a discrete set of time instants t_i, where i = 1, 2, ... n, that a computer computes values of some measurable function of the orbital parameters and time $f_c (a_1, \ldots a_M, t)$. We further assume that the actually observed or computed function may be written as the sum of 3 terms in the following way:

$$f_m(t) = f_c (t) + b + n (t) \tag{1}$$

The term b is a constant bias error, n (t) is assumed to be a sample from a zero mean stationary gaussian noise process whose correlation time is short with respect to the time interval between successive samples. $f_c(t)$ then represents the data if there were no noise or errors corrupting the observations.

Equation (1) may be rearranged to yield

$$n (t) = f_m(t) - f_c(t) - b \tag{2}$$

Since n (t) is an additive uncorrelated zero mean normal noise process, the likelihood function for N observations made at times $t_i, \ldots t_N$ may be written:

$$L = \frac{1}{(2\pi)^{\frac{N}{2}} \sigma^N} \exp\left[-\frac{1}{2\sigma^2} \sum_{i=1}^{N} n^2(i)\right] \tag{3}$$

where σ^2 is the variance of a noise sample and where we have written n (i) instead of $n(t_i)$ for convenience.

Equivalently from Equation (2) we may write:

$$L = \frac{1}{\sigma^N (2\pi)^{\frac{N}{2}}} \exp\left\{-\frac{1}{2\sigma^2} \sum_{i=1}^{N} \left[f_m(i) - f_c(i) - b\right]^2\right\}. \tag{4}$$

Now the estimators $\hat{a}_1, \ldots \hat{a}_m$ for the M orbital parameters are those functions of the observed data which render the data most probable. That is,we estimate the orbit parameters by those functions of the observed data which maximize the function L. These estimators are the maximum likelihood estimators for the parameters and are themselves statistics having certain probability distributions, determined by the distribution of the error process as well as the functional dependence of the measurable quantity upon the orbital parameters $a_1, \ldots a_M$. To determine how accurately these parameters can be measured we might wish to compute the mean square error of the estimators. If the estimators are unbiased as maximum likelihood estimators must be for sufficiently large smoothing times, then the appropriate measure of the accuracy of the estimator is its variance, so that the quantities of interest would be the variances of the estimators.

However, the estimators of the orbital parameters are all computed as functions of the same observed data so that generally, the probability distributions of these estimators will not be independent. Thus, the errors in the estimators may be expected to be correlated. It is important to note that the correlation which we are discussing here is the correlation among the estimator errors which arises from the fact that the parametric estimators are computed from the same observed data, and that this sort of correlation will,in general, exist even though the successive noise samples are uncorrelated.

Now since it is likely that the ultimate use to which the present analysis will be put will be the determination of certain system errors as functions of the estimator errors, and since these system errors will depend in general upon the correlation existing between the estimator errors, then it is important to compute these correlations. Thus, because of the foregoing remarks, instead of merely computing the variance of the estimators of the orbital parameters we shall compute the covariance matrix of the estimators which will simultaneously provide the information needed about the estimator variances and the correlation between the estimators.

Let Cov $(\hat{a}_i, \hat{a}_j)$ denote the covariance matrix of the maximum likelihood estimators of the orbit parameters, where the i, j th element of this matrix is the covariance between the estimators of the i th and the j th orbit parameters.

Since the maximum likelihood estimators of the parameters are those functions of the observed data which maximize the likelihood function, it follows from Equation(4) the estimators of the parameters $\hat{a}_1, \ldots \hat{a}_M$ and the estimator of the bias error $\hat{b}$ are obtained by the simultaneous solution of the M + 1 equations.

$$\frac{1}{N} \sum_{i=1}^{N} \left[f_m (i) - f_c (i) - b \right] \frac{\partial f_c(i)}{\partial a_k} = 0 \quad \text{for } k = 1, \ldots, M \tag{5}$$

and

$$\frac{1}{N} \sum_{i=1} \left[f_m (i) - f_c (\hat{a}_1, \ldots \hat{a}_M, i) \right] = \hat{b} \tag{6}$$

That is, the estimators $\hat{a}_1, \ldots \hat{a}_M$ are obtained by solving the system (5) for $a_1, \ldots, a_M$ as functions of the observed data. The bias error is then estimated as in (6) by the arithmetic average of the difference between the observed value and the values obtained by substituting the estimators in place of the parametric values.

The system (5) is, in general, a nonlinear system of algebraic or transcendental equations and will often be incapable of yielding an exact solution. However, for a sufficient number of independent observations or equivalently for a sufficiently large smoothing time, the estimators $\hat{a}_1, \ldots, \hat{a}_M$ and $\hat{b}$ may be written:

$$\left.\begin{aligned} \hat{a}_i &= a_i + \Delta a_i \\ \hat{b} &= b + \Delta b \end{aligned}\right\} \qquad (7)$$

where the random error terms Δa_i and Δb are negligible except where they appear as first order terms. Qualitatively this says that our estimates get good when we have enough independent data. Using (7) we may write:

$$f_c\,(\hat{a}_1, \ldots \hat{a}_M, i) = f_c\,(a_1, \ldots, a_M, i) + \sum_{k=1}^{M} \frac{\partial f_c}{\partial a_k}\, \Delta a_k \qquad (8)$$

Substituting Equations (7) and (8) into Equations (5) and (6) allows us to write the following system of M + 1 equations which are linear in the random error quantities Δa_i and Δ_b.

$$C\, \Delta a = e\,. \qquad (10)$$

For notational convenience, we have employed vector-matrix notation in writing Equation (10). Δa and e are M + 1 dimensional column vectors and C is an M + 1 by M + 1 square matrix where the components of these matrices are as defined below.

$$\Delta a = \begin{pmatrix} \Delta a_1 \\ \vdots \\ \vdots \\ \Delta a_M \\ \Delta b \end{pmatrix} \qquad (11)$$

$$e = \begin{pmatrix} \displaystyle\sum_{i=1}^{N} \frac{\partial f_c(i)}{\partial a_1}\, n\,(i) \\ \vdots \\ \vdots \\ \displaystyle\sum_{i=1}^{N} \frac{\partial f_c(i)}{\partial a_M}\, n\,(i) \\ \displaystyle\sum_{i=1}^{N} n\,(i) \end{pmatrix} \qquad (12)$$

$$C_{ij} = \sum_{k=1}^{N} \frac{\partial f_c}{\partial a_i}(k) \; \frac{\partial f_c}{\partial a_j}(k) \qquad 1 \le i, j \le M$$

$$C_{M+1,j} = \sum_{k=1}^{N} \frac{\partial f_{c(k)}}{\partial a_j} = C_{j,M+1} \qquad 1 \le j \le M \tag{13}$$

$$C_{M+1,\,M+1} = N$$

Now Equation (10) may be solved for the error vector in the estimator components by writing:

$$\Delta a = C^{-1} e \tag{14}$$

Taking the transpose of both sides of (14) yields

$$(\Delta a)^T = e^T (C^{-1})^T = e^T C^{-1}. \tag{15}$$

In (15), the superscript T denotes the transposition operation and we have also used the fact that the matrix C is symmetric.

Multiplying Equation (15) on the left by Equation (14) yields the following matrix equation:

$$\Delta a \, (\Delta a)^T = C^{-1} e \; e^T C^{-1} \tag{16}$$

where we note that Equation (16) is a relationship between two square matrices rather than between two column vectors.

Now by definition the covariance matrix of the parametric estimators is the expected value of the matrix $\Delta a \, (\Delta a)^T$, thus, if we let E denote the expected value operator we obtain:

$$\operatorname{Cov}(\hat{a}_i, \hat{a}_j) = C^{-1} E\left[e \; e^T \right] C^{-1} \tag{17}$$

However, since n (i) is an additive, stationary white noise process, then,

$$E\left[n(i) \, n(j) \right] = \delta_{ij} \, \sigma^2 \tag{18}$$

where δ_{ij} is the Kronecker delta. Equation (18) together with Equations (12) and (13) imply that

$$E\left[e\ e^T\right] = \sigma^2\ C. \tag{19}$$

Thus, the expression for the covariance matrix of the estimators of the orbit parameters may be written in the following simple form:

$$\mathrm{Cov}\ (\hat{a}_i,\ \hat{a}_j) = \sigma^2\ C^{-1} \tag{20}$$

The utility of the expression (20) for the covariance matrix of the estimators may be illustrated as follows. Let ψ be an arbitrary function whose value depends upon the orbit parameters and possibly time and other non-random quantites whose values are known and suppose that we desire the error in ψ due to the errors in the estimators. Then if $\frac{\partial \psi}{\partial a}$ denotes the M + 1 dimensional column vector whose first M components are $\frac{\partial \psi}{\partial a_i}$ $i = 1, \ldots M$ and whose (M + 1) st component is $\frac{\partial \psi}{\partial b}$, we may immediately write the following expression for the variance of ψ.

$$\sigma_\psi^2 = \sigma^2 \left(\frac{\partial \psi}{\partial a}\right)^T C^{-1} \left(\frac{\partial \psi}{\partial a}\right) \tag{21}$$

Equation (21) automatically accounts for the effect of any correlation which might exist between the estimators and explicitly exhibits the reciprocal dependence between the variance and the number of independent pieces of data in a concise manner.

APOLLO NOTE NO. 77

J. Holdsworth
19 June 1963

CALCULATION OF COVARIANCE MATRICES FOR MULTIPLE UNCORRELATED DATA SOURCES

The purpose of this note is to extend the methods developed in Apollo Notes Nos. 3 and 43 to the case where multiple data inputs are available. The previous notes have covered the case where only one form of data, such as range rate from Doppler measurements, was available. The present note will extend these procedures to the case where range, range rate, and angular data are all available. In the subsequent analysis we shall assume that there is no autocorrelation in any of the three data inputs and that the different types of data are not cross correlated.

As before we shall assume that there are certain parameters a_i, $i = 1, \cdots 6$, which we wish to estimate on the basis of our observed data. We shall also assume that we have available range, range rate, and angular data which we shall denote by $R, \dot{R}$ and θ respectively and that $R, \dot{R}$ and θ may be written as invertible functions of the parameters of interest a_i.

If our measurements could be made with complete accuracy then any six observations would theoretically suffice to allow us to determine the parameters in question. However, the measured data is corrupted by random noise hence we must use our data to obtain estimators $\hat{a}_i$ of the orbit parameters a_i. In reality we are more interested in this note in obtaining an expression for the accuracy or asymptotic accuracy with which the parameters can be estimated rather than the computation of the estimators from the observed data.

As in Apollo Notes Nos. 3 and 43 we assume that we may write the following expressions:

$$R_m = R_c(a_i, t) + n_R(t)$$

$$\dot{R}_m = \dot{R}_c(a_i, t) + n_{\dot{R}}(t) \qquad (1)$$

$$\theta_m = \theta_c(a_i, t) + n_\theta(t)$$

In equations (1) the quantities subscripted m are the measured quantities, the quantities subscripted c refer to the correct functional values if the data were not contaminated by noise and $n_R(t)$, $n_{\dot{R}}(t)$, $n_\theta(t)$ indicates the additive random noise.

As mentioned before we shall assume that the noise processes $n_R(t)$, $n_{\dot{R}}(t)$ and $n_\theta(t)$ are independent, i.e. not cross correlated, zero mean stationary white gaussian processes with variances σ_R^2, $\sigma_{\dot{R}}^2$ and σ_θ^2, respectively. From equation (1) we see that if we have N observations on R, $\dot{R}$ and θ that we may write the following expression for the likelihood function of the data.

$$L = \frac{1}{(2\pi)^{3N/2} \sigma_R^N \sigma_{\dot{R}}^N \sigma_\theta^N} \exp -\mathcal{L} \qquad (2)$$

where:

$$\mathcal{L} = \frac{1}{2\sigma_R^2} \sum_{k=1}^{N} \left[R_m(k) - R_c(a_i, k) \right]^2$$

$$+ \frac{1}{2\sigma_{\dot{R}}^2} \sum_{k=1}^{N} \left[\dot{R}_m(k) - \dot{R}_c(a_i, k) \right]^2$$

$$+ \frac{1}{2\sigma_\theta^2} \sum_{k=1}^{N} \left[\theta_m(k) - \theta_c(a_i, k) \right]^2 \qquad (3)$$

As before the maximum likelihood estimators a_i of the parameters a_i are those functions of the data which make the data most probable or maximize the value of the likelihood function. They are obtained by solving the equations

$$\frac{\partial \mathscr{L}}{\partial a_i} = 0 \qquad i = 1, \cdots 6 \tag{4}$$

for the parameters a_i as functions of the observed data R_m, $\dot{R}_m$ and θ_m.

Performing the indicated differentiations and substituting into equation (4) we obtain:

$$\frac{1}{\sigma_R^2} \sum_{k=1}^{N} \left[R_c(a_i, k) - R_m(k)\right] \frac{\partial R_c}{\partial a_i} + \frac{1}{\sigma_{\dot{R}}^2} \sum_{k=1}^{N} \left[\dot{R}_c(a_i, k) - \dot{R}_m(k)\right] \frac{\partial \dot{R}_c}{\partial a_i}$$

$$+ \frac{1}{\sigma_R^2} \sum_{k=1}^{N} \left[\theta_c(a_i, k) - \theta_m(k)\right] \frac{\partial \theta_c}{\partial a_i} = 0 \qquad \text{for } i = 1, \cdots 6 \tag{5}$$

Now the functional forms R_c, $\dot{R}_c$, θ_c are assumed known, as are the observed data points $R_m(k)$, $\dot{R}_m(k)$, $\theta_m(k)$, thus equation (5) is actually a system of 6 equations in the a_i which may be solved as a function of the known data. The solutions $\hat{a}_i$ of this system of equations in terms of the observed data are the maximum likelihood estimators of the orbital parameters a_i.

Now as we have done in the previous notes we shall assume that the smoothing time or number of samples N is sufficiently large so that the following expressions may be written for the maximum likelihood estimators:

$$\hat{a}_i = a_i + \Delta a_i \tag{6}$$

where the Δa_i are sufficiently small so that only first order terms in these random perturbations need be retained in various series expansions. This assumption is always valid for sufficiently large sample sizes and is exactly true whenever the dependence of the functions f_c on the a_i parameters is linear.

Utilizing the above assumptions we may write the following approximate expressions.

$$R_c(\hat{a}_i, k) = R_c(a_i, k) + \sum_{i=1}^{6} \frac{\partial R_c}{\partial a_i}(a_i, k)\Delta a_i$$

$$\dot{R}_c(\hat{a}_i, k) = \dot{R}_c(a_i, k) + \sum_{i=1}^{6} \frac{\partial \dot{R}_c}{\partial a_i}(a_i, k)\Delta a_i \qquad (7)$$

$$\theta_c(\hat{a}_i, k) = \theta_c(a_i, k) + \sum_{i=1}^{6} \frac{\partial \theta}{\partial a_i}(a_i, k)\Delta a_i$$

$$\frac{\partial R_c}{\partial a_i}(\hat{a}_i, k) \approx \frac{\partial R_c}{\partial a_i}(a_i, k), \quad \frac{\partial \dot{R}_c}{\partial a_i}(\hat{a}_i, k) = \frac{\partial \dot{R}_c}{\partial a_i}(a_i, k), \frac{\partial \theta}{\partial a_i}(\hat{a}_i, k) = \frac{\partial \theta}{\partial a_i}(a_i, k)$$

Substituting equation (7) into equation (5) and performing some routine algebra yields the following system of equations which is linear in the random perturbation quantities Δa_i.

$$\sum_{j=1}^{6} \left\{ \sum_{k=1}^{N} \left(\frac{1}{\sigma_R^2} \frac{\partial R_c}{\partial a_i}(k) \frac{\partial R_c}{\partial a_j}(k) + \frac{1}{\sigma_{\dot{R}}^2} \frac{\partial \dot{R}_c}{\partial a_i}(k) \frac{\partial \dot{R}_c}{\partial a_j}(k) + \frac{1}{\sigma_\theta^2} \frac{\partial \theta_c}{\partial a_i}(k) \frac{\partial \theta_c}{\partial a_j}(k) \right) \right\} \Delta a_j$$

$$= \sum_{k=1}^{N} \left[\frac{1}{\sigma_R^2} \frac{\partial R_c}{\partial a_i}(k) \left\{ R_m(k) - R_c(k) \right\} + \frac{1}{\sigma_{\dot{R}}^2} \frac{\partial \dot{R}_c}{\partial a_i}(k) \left\{ \dot{R}_m(k) - R_c(k) \right\} \right.$$

$$\left. + \frac{1}{\sigma_\theta^2} \frac{\partial \theta_c}{\partial a_i}(k) \left\{ \theta_m(k) - \theta_c(k) \right\} \right] \qquad \text{for } i = 1, 2, \cdots 6 \qquad (8)$$

In equation (8) it is perhaps worth mentioning that $R_c(k)$, $\dot{R}_c(k)$, $\theta_c(k)$ refer to the nominal value of these quantities at the time of the k^{th} observation, while $R_m(k)$, $\dot{R}_m(k)$, $\theta_m(k)$ refer to the observed data obtained from the k^{th} observation.

Again if we define the estimator error vector Δa by the column matrix

$$\Delta a = \begin{pmatrix} \Delta a_1 \\ \Delta a_2 \\ \Delta a_3 \\ \Delta a_4 \\ \Delta a_5 \\ \Delta a_6 \end{pmatrix} \qquad (9)$$

then using matrix notation we may write the following vector equation:

$$C \Delta a = e. \qquad (10)$$

In equation (10) C is a 6×6 matrix whose i, j^{th} element is given by:

$$C_{ij} = \sum_{k=1}^{N} \left[\frac{1}{\sigma_R^2} \frac{\partial R_c}{\partial a_i}(k) \frac{\partial R_c}{\partial a_j}(k) + \frac{1}{\sigma_{\dot{R}}^2} \frac{\partial \dot{R}_c}{\partial a_i}(k) \frac{\partial \dot{R}_c}{\partial a_j}(k) + \frac{1}{\sigma_\theta^2} \frac{\partial \theta_c}{\partial a_i}(k) \frac{\partial \theta_c}{\partial a_j}(k) \right]$$

(11)

Similarly e is a column vector whose i^{th} component is given by:

$$e_i = \sum_{k=1}^{N} \left[\frac{1}{\sigma_R^2} \frac{\partial R_c}{\partial a_i}(k) \left(R_m(k) - R_c(k)\right) + \frac{1}{\sigma_{\dot{R}}^2} \frac{\partial \dot{R}_c}{\partial a_i}(k) \left(\dot{R}_m(k) - \dot{R}_c(k)\right) + \frac{1}{\sigma_\theta^2} \frac{\partial \theta_c}{\partial a_i}(k)\left(\theta_m(k) - \theta_c(k)\right) \right]$$

for $i = 1, 2, \cdots 6$ (12)

Now, if the joint distribution of the parametric estimators is non-singular; i. e., if the total probability mass of the estimator distribution does not lie in some subspace of dimension 5 or lower, then equation (10) may be formally solved to yield:

$$\Delta a = C^{-1} e \quad (13)$$

Taking the transpose of both sides of equation (13) and noting that $C_{ij} = C_{ji}$, we obtain:

$$\Delta a^T = e^T C^{-1} \quad (14)$$

Equations (13) and (14) are vector equations. Multiplication of (13) on the right by equation (14) yields the matrix equation

$$\Delta a \, \Delta a^T = C^{-1} e \, e^T C^{-1} \quad (15)$$

Equation (15) is a matrix rather than a vector equation, i. e., the quantities on both sides of equation (15) are 6 x 6 matrices. Moreover, the elements of these matrices are random so that in similar observations over the same smoothing interval we would expect a random variation in the elements.

Since we have assumed large smoothing times the estimators $\hat{a}_i$ of the parameters a_i may be assumed to be unbiased, i. e.

$$E\,\hat{a}_i = a_i \tag{16}$$

or equivalently

$$E\Delta a_i = 0 \tag{17}$$

- a fact which is always asymptotically true for maximum likelihood estimators.

Since the estimators are unbiased the covariance matrix of the estimator errors is obtained by taking the expected value of both sides of equation (15) with respect to the joint distribution of the noise processes. Thus, we may write:

$$\mathrm{Cov}\,(\hat{a}_i, \hat{a}_j) = E\left[\Delta a \,\Delta a^T\right] = C^{-1}\,E(e\,e^T)\,C^{-1} \tag{18}$$

where E denotes the expected value operator.

Since the matrix C is known and assumed non singular, it follows from equation (18) that we have an expression for our desired covariance matrix once we have computed

$$E\,(e\,e^T) \tag{19}$$

The element in the i, jth position of the matrix $e\,e^T$ is simply $e_i e_j$, where the expression for e_i and e_j is given by equation (12). However, comparison with equation (1) shows that we may write:

$$e_i = \sum_{k=1}^{N}\left[\frac{1}{\sigma_R^2}\frac{\partial R_c}{\partial a_i}(k)\,n_R(k) + \frac{1}{\sigma_{\dot{R}}^2}\frac{\partial \dot{R}_c}{\partial a_i}(k)\,n_{\dot{R}}(k) + \frac{1}{\sigma_\theta^2}\frac{\partial \theta}{\partial a_i}(k)\,n_\theta(k)\right] \tag{20}$$

Now a term on the main diagonal of $e\,e^T$ is of the form e_i^2 where e_i is given by equation (20). Since $n_R(k)$, $n_{\dot{R}}(k)$, $n_\theta(k)$ were assumed to be independent zero mean stationary gaussian random processes then we have the following expressions:

$$
\begin{aligned}
E\left[n_R(k)\, n_R(\ell)\right] &= \delta_{k\ell}\,\sigma_R^2 \\
E\left[n_{\dot{R}}(k)\, n_{\dot{R}}(\ell)\right] &= \delta_{k\ell}\,\sigma_{\dot{R}}^2 \\
E\left[n_\theta(k)\, n_\theta(\ell)\right] &= \delta_{k\ell}\,\sigma_\theta^2
\end{aligned}
\tag{21}
$$

and

$$E\left[n_R(k)\, n_{\dot{R}}(\ell)\right] = 0 = E\left[n_R(k)\, n_\theta(\ell)\right] = E\left[n_{\dot{R}}(k)\, n_\theta(\ell)\right]$$

for all k and ℓ, where $\delta_{k\ell}$ is the Kronecker delta function.

Using equations (20) and (21) we see that a main diagonal term of the matrix $E\left[e\,e^T\right]$ is given by:

$$E\; e_{ii} = E\; e_i^2 = \sum_{k=1}^{N} \left[\left(\frac{\partial R_c(k)}{\partial a_i}\right)^2 \frac{1}{\sigma_R^2} + \left(\frac{\partial \dot{R}_c(k)}{\partial a_i}\right)^2 \frac{1}{\sigma_{\dot{R}}^2} + \left(\frac{\partial \theta(k)}{\partial a_i}\right)^2 \frac{1}{\sigma_\theta^2} \right] \tag{22}$$

for i = 1, 2, $\cdots$ 6. Thus, we have an expression for the diagonal elements of $E\left[e\,e^T\right]$.

Now consider an off diagonal element $E\left[e_i\, e_j\right]$ where $i \neq j$. Then again from equations (20) and (21) we may write:

$$
\begin{aligned}
E\left[e_{ij}\right] = E\left[e_i\, e_j\right] = \sum_{k=1}^{N} \Bigg[& \frac{1}{\sigma_R^2} \frac{\partial R_c(k)}{\partial a_i} \frac{\partial R_c(k)}{\partial a_j} \\
& + \frac{1}{\sigma_{\dot{R}}^2} \frac{\partial \dot{R}_c(k)}{\partial a_i} \frac{\partial \dot{R}_c(k)}{\partial a_j} + \frac{1}{\sigma_\theta^2} \frac{\partial \theta_c(k)}{\partial a_i} \frac{\partial \theta_c(k)}{\partial a_j} \Bigg]
\end{aligned}
\tag{23}
$$

Now from equations (22) and (23) we may explicitly calculate the elements of the matrix $E\left[e\, e^T\right]$. Also since the elements of the matrix C are known then C^{-1} is known also which gives us our desired covariance matrix by substitution into equation (18).

However, comparison of the defining equations for the elements of the matrices C and $E\left[e\, e^T\right]$ - i.e., equations (11), (22) and (23) - reveal the interesting fact that

$$E\left[e\, e^T\right] = C. \tag{24}$$

Thus equation (24) allows us to write the following expression for the covariance matrix of the errors in the estimators of the orbital parameters.

$$\mathrm{Cov}\,(\hat{a}_i, \hat{a}_j) = C^{-1} \tag{25}$$

CONCLUSION

In this note it will be noticed that there was no systematic bias error assumed in any of the three data inputs. In reality there is reason to suspect that there may be a non-negligible systematic bias error in the range and angle data. The extension of this analysis to cover that situation is straightforward and is more of a notational nuisance than a conceptual difficulty. The interested reader should be able to make the necessary amendments by using the analysis in either Notes 43 or 3 as a guide.

A more serious shortcoming of this note is that using the JPL measuring procedure the range and the range rate data are very strongly cross correlated. As other data collection schemes are under consideration which would probably tend to reduce this cross correlation, it is hoped that the results obtained in this note may be of use in some cases of genuine physical interest.

Since the cross correlation of the data inputs does not unduly encumber the necessary mathematics as long as the individual error processes may be assumed to be white or uncorrelated, another note will appear shortly extending the current results to the case where appreciable cross correlation exists between the data inputs.

APOLLO NOTE NO. 82

H. Engel
16 July 1963

CALCULATION OF COVARIANCE MATRICES III

This note presents a means for finding the covariance matrix when Doppler observations of a vehicle in an elliptic or circular orbit about a moving Moon are made from a DSIF or MSFN facility on the surface of the Earth. The orbit of the Moon is assumed circular, but an elliptical orbit could be used with just slightly more computation.

The parameters chosen to describe the vehicle orbit are its position and velocity with respect to the Moon at the time of the first observation. This choice of parameters has the advantage that it is possible to obtain expressions for the partial derivatives used in calculating the inverse of the covariance matrix that do not blow up when the orbit eccentricity approaches or equals zero.

The vector from the Moon to the vehicle is $\bar{r}$, the vector from the Earth to the Moon is $\bar{X}_m$, and the vector from the center of the Earth to the observing station is $\bar{X}_d$. The vector from the observing station to the vehicle is $\bar{s}$,

$$\bar{s} = \bar{X}_m + \bar{r} - \bar{X}_d$$

The observed quantity is the rate of change of distance between the observing station and the vehicle, i.e., $\dot{s}$. Now,

$$\dot{\bar{s}} = \dot{\bar{X}}_m + \dot{\bar{r}} - \dot{\bar{X}}_d$$

and

$$\dot{s} = \left(\frac{\bar{s}}{s}\right) \cdot \dot{\bar{s}}$$

Then,

$$\frac{\partial \dot{\bar{s}}}{\partial a_j} = \frac{\partial \dot{\bar{X}}_m}{\partial a_j} + \frac{\partial \dot{\bar{r}}}{\partial a_j} - \frac{\partial \dot{\bar{X}}_d}{\partial a_j}$$

$$= \frac{\partial \dot{\bar{r}}}{\partial a_j}$$

and

$$\frac{\partial \dot{s}}{\partial a_j} = \frac{1}{s}\left[\bar{s} \cdot \frac{\partial \dot{\bar{s}}}{\partial a_j} + \dot{\bar{s}} \cdot \frac{\partial \bar{s}}{\partial a_j} - \dot{s}\;\frac{\partial s}{\partial a_j}\right]$$

$$= \frac{1}{s}\left[\bar{s} \cdot \frac{\partial \dot{\bar{s}}}{\partial a_j} + \dot{\bar{s}} \cdot \frac{\partial \bar{s}}{\partial a_j} - \frac{\dot{s}}{s}\;\bar{s} \cdot \frac{\partial \bar{s}}{\partial a_j}\right]$$

$$= \frac{1}{s}\left[\bar{s} \cdot \frac{\partial \dot{\bar{r}}}{\partial a_j} + \left(\dot{\bar{s}} - \frac{\dot{s}}{s}\;\bar{s}\right) \frac{\partial \bar{r}}{\partial a_j}\right]$$

Terms of the form $\sum \frac{\partial \dot{s}_i}{\partial a_j}\;\frac{\partial \dot{s}_i}{\partial a_k}$ are used in computing the matrix from which the covariance matrix is found.

It is convenient to use different co-ordinate systems in computing $\overline{X}_m$, $\bar{r}$, and $\overline{X}_d$. By co-ordinate rotations $\overline{X}_m$ and $\overline{X}_d$ are expressed in the same co-ordinates as $\bar{r}$ so $\partial \dot{s}/\partial a_j$ can be evaluated.

In the work that follows, the various quantities that must be employed are presented in the sequence in which they are used in actual computations.

The x'y'z' co-ordinate system is right-handed, non-rotating and Moon-centered. x' is directed along the initial position vector $(x_o', 0, 0)$ of the vehicle, vehicle motion is in the x'y' - plane, and y' is directed so that $\dot{y}_o'$ is positive. The orbit parameters are the components of $\bar{r}_o$ and $\dot{\bar{r}}_o$ in the x'y'z' co-ordinate system; these are x_o', y_o', z_o', $\dot{x}_o'$, $\dot{y}_o'$, and $\dot{z}_o'$, and are called a_1 through a_6 respectively.

Note well that one can not employ the fact that a_2, a_3 and a_6 are zero, until after derivatives are taken; otherwise incorrect results are obtained. For this reason two different expressions for a quantity may be found, one perhaps including a_3, a_5, and a_6, and used for differentiation, and a second expression without a_3, a_5, and a_6 and used for computation. The quantities used for computation are enclosed in boxes.

The initial radius $\overline{r}_o$ is given by:

$$\overline{r}_o = a_1 \hat{x}' + a_2 \hat{y}' + a_3 \hat{z}'$$

in which the caret denotes a unit vector.

$$r_o^2 = a_1^2 + a_2^2 + a_3^2$$

$$2 r_o \frac{\partial r_o}{\partial a_j} = 2 a_1 \delta_{1j} + 2 a_2 \delta_{2j} + 2 a_3 \delta_{3j}$$

$$\boxed{\frac{\partial r_o}{\partial a_j} = \delta_{1j}}$$

in which $\delta_{i,j}$ is the Kronecker delta

$$\delta_{i,j} = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{cases}$$

$$r_o = a_1$$

$$\dot{\overline{r}}_o = a_4 \hat{x}' + a_5 \hat{y}' + a_6 \hat{z}'$$

$$r_o^2 = \overline{r}_o \cdot \overline{r}_o$$

$$2 r_o \dot{r}_o = 2 \overline{r}_o \cdot \dot{\overline{r}}_o$$

$$\dot{r}_o = \frac{\overline{r}_o \cdot \dot{\overline{r}}_o}{r_o}$$

$$= \frac{a_1 a_4 + a_2 a_5 + a_3 a_6}{r_o}$$

$$\dot{r}_o = a_4$$

$$\frac{\partial (r_o \dot{r}_o)}{\partial a_j} = r_o \frac{\partial \dot{r}_o}{\partial a_j} + \dot{r}_o \frac{\partial r_o}{\partial a_j} = \frac{\partial}{\partial a_j} (a_1 a_4 + a_2 a_5 + a_3 a_6)$$

$$r_o \frac{\partial \dot{r}_o}{\partial a_j} = \delta_{1j} a_4 + \delta_{2j} a_5 + \delta_{3j} a_6 + \delta_{4j} a_1 + \delta_{5j} a_2 + \delta_{6j} a_3 - \dot{r}_o \frac{\partial r_o}{\partial a_j}$$

$$\boxed{a_1 \frac{\partial \dot{r}_o}{\partial a_j} = \delta_{4j} a_1 + \delta_{2j} a_5}$$

The angular momentum H is given by:

$$\overline{H} = (a_2 a_6 - a_3 a_5) \hat{x}' + (a_3 a_4 - a_1 a_6) \hat{y}' + (a_1 a_5 - a_2 a_4) \hat{z}'$$

$$H^2 = (a_2 a_6 - a_3 a_5)^2 + (a_3 a_4 - a_1 a_6)^2 + (a_1 a_5 - a_2 a_4)^2$$

$$2 H \frac{\partial H}{\partial a_j} = 2 (a_2 a_6 - a_3 a_5) \frac{\partial (a_2 a_6 - a_3 a_5)}{\partial a_j}$$

$$+ 2 (a_3 a_4 - a_1 a_6) \frac{\partial (a_3 a_4 - a_1 a_6)}{\partial a_j}$$

$$+ 2 (a_1 a_5 - a_2 a_4) \frac{\partial (a_1 a_5 - a_2 a_4)}{\partial a_j}$$

$$\boxed{H = a_1 a_5}$$

$$\boxed{\frac{\partial H}{\partial a_j} = \delta_{1j} a_5 - \delta_{2j} a_4 + \delta_{5j} a_1}$$

The orbit energy E is given by:

$$2E = -\frac{2\mu}{r_o} + a_4^2 + a_5^2 + a_6^2$$

$$\boxed{2E = -\frac{2\mu}{a_1} + a_4^2 + a_5^2}$$

$$2\frac{\partial E}{\partial a_j} = 2\frac{\mu}{r_o^2}\frac{\partial r_o}{\partial a_j} + 2a_4\delta_{4j} + 2a_5\delta_{5j} + 2a_6\delta_{6j}$$

$$\boxed{\frac{\partial E}{\partial a_j} = \delta_{1j}\frac{\mu}{a_1^2} + \delta_{4j} a_4 + \delta_{5j} a_5}$$

If the orbit eccentricity is e, then,

$$r = \frac{H^2}{\mu(1 + e\cos\theta)}$$

and

$$\dot{r} = \frac{\mu e}{H}\sin\theta$$

where θ is the central angle between the vehicle and perilune. Then

$$e\cos\theta = \frac{H^2}{\mu r} - 1$$

$$e\cos\theta_o = \frac{H^2}{\mu r_o} - 1$$

$$\boxed{e\cos\theta_o = \frac{H a_5}{\mu} - 1}$$

and

$$e \sin \theta = \frac{H \dot{r}}{\mu}$$

$$e \sin \theta_o = \frac{H \dot{r}_o}{\mu}$$

$$\boxed{e \sin \theta_o = \frac{H a_4}{\mu}}$$

Then,

$$\boxed{e^2 = (e \cos \theta_o)^2 + (e \sin \theta_o)^2} \quad \boxed{e = \sqrt{e^2}}$$

Also,

$$e^2 = 1 + \frac{2EH^2}{\mu^2}$$

so

$$\boxed{1 - e^2 = - 2 E \left(\frac{H}{\mu}\right)^2}$$

and

$$\boxed{(1 - e^2)^{1/2} = \sqrt{- 2 E} \left(\frac{H}{\mu}\right)}$$

$$\frac{\partial e^2}{\partial a_j} = \frac{2}{\mu^2} \left[H^2 \frac{\partial E}{\partial a_j} + E \frac{\partial H^2}{\partial a_j} \right]$$

$$\boxed{\frac{\partial e^2}{\partial a_j} = 2 \left(\frac{H}{\mu}\right)^2 \left[\frac{\partial E}{\partial a_j} + \frac{2E}{H} \frac{\partial H}{\partial a_j}\right]}$$

If the eccentricity is not zero, then θ_o can be found from:

$$\boxed{\theta_o = \tan^{-1} \left(\frac{e \sin \theta_o}{e \cos \theta_o} \right)}$$

and if the eccentricity is zero, θ_o can be arbitrarily chosen as zero. θ_o is always between $-\pi$ and π. In both cases $\sin \theta_o$ and $\cos \theta_o$ are evaluated from θ_o.

Now,

$$\frac{\partial (e \cos \theta_o)}{\partial a_j} = \frac{\partial \left(\frac{H^2}{\mu r_o} - 1 \right)}{\partial a_j}$$

$$= \frac{1}{\mu} \left[r_o^{-1} \frac{\partial H^2}{\partial a_j} + H^2 \frac{\partial r_o^{-1}}{\partial a_j} \right]$$

$$= \frac{H}{\mu r_o} \left[2 \frac{\partial H}{\partial a_j} - \frac{H}{r_o} \frac{\partial r_o}{\partial a_j} \right]$$

$$\boxed{\frac{\partial (e \cos \theta_o)}{\partial a_j} = \frac{a_5}{\mu} \left[2 \frac{\partial H}{\partial a_j} - \delta_{1j} a_5 \right]}$$

and

$$\frac{\partial (e \sin \theta_o)}{\partial a_j} = \frac{\partial \left(\frac{H \dot{r}_o}{\mu} \right)}{\partial a_j}$$

$$= \frac{1}{\mu} \left[H \frac{\partial \dot{r}_o}{\partial a_j} + \dot{r}_o \frac{\partial H}{\partial a_j} \right]$$

$$\boxed{\frac{\partial (e \sin \theta_o)}{\partial a_j} = \frac{a_5}{\mu}\left(a_1 \frac{\partial \dot{r}_o}{\partial a_j}\right) + \frac{a_4}{\mu} \frac{\partial H}{\partial a_j}}$$

The sine and cosine of the initial eccentric anomaly $\mathcal{E}_o$ are given by:

$$\sin \mathcal{E}_o = \frac{(1 - e^2)^{1/2} \sin \theta_o}{1 + e \cos \theta_o}$$

$$\boxed{\sin \mathcal{E}_o = \frac{\sqrt{-2E}}{a_5} \sin \theta_o}$$

and

$$\cos \mathcal{E}_o = \frac{e + \cos \theta_o}{1 + e \cos \theta_o}$$

$$\boxed{\cos \mathcal{E}_o = \frac{e + \cos \theta_o}{\left(\frac{Ha_5}{\mu}\right)}}$$

$$\boxed{\mathcal{E}_o = \tan^{-1}\left(\frac{\sin \mathcal{E}_o}{\cos \mathcal{E}_o}\right)}$$

$\mathcal{E}_o$ is always between $-\pi$ and π. The mean anomaly at the initial condition is:

$$\boxed{M_o = \mathcal{E}_o - e \sin \mathcal{E}_o}$$

and the mean motion, n, is

$$\boxed{n = \frac{-2E\sqrt{-2E}}{\mu}}$$

so the time, t_o, from perilune to the initial point, T = 0, is

$$\boxed{t_o = M_o/n}$$

also,

$$\frac{\partial n}{\partial a_j} = \frac{3\,n}{2E}\;\frac{\partial E}{\partial a_j}$$

Then,

$$e \sin \mathcal{E}_o = \frac{(1-e^2)^{1/2}\,(e \sin \theta_o)}{1 + e \cos \theta_o}$$

$$\frac{1}{e \sin \mathcal{E}_o} \cdot \frac{\partial (e \sin \mathcal{E}_o)}{\partial a_j} = \frac{\partial\,(1-e^2)^{1/2} / \partial a_j}{(1-e^2)^{1/2}} + \frac{\partial (e \sin \theta_o)/\partial a_j}{e \sin \theta_o}$$

$$- \frac{\partial\,(e \cos \theta_o)/\partial a_j}{1 + e \cos \theta_o}$$

$$\frac{\partial\,(e \sin \mathcal{E}_o)}{\partial a_j} = \frac{-e \sin \mathcal{E}_o}{2\,(1-e^2)}\;\frac{\partial e^2}{\partial a_j} + \frac{\sin \mathcal{E}_o}{\sin \theta_o}\;\frac{\partial\,(e \sin \theta_o)}{\partial a_j} - \frac{e \sin \mathcal{E}_o}{1 + e \cos \theta_o}\;\frac{\partial (e \cos \theta_o)}{\partial a_j}$$

$$\boxed{\frac{\partial\,(e \sin \mathcal{E}_o)}{\partial a_j} = \frac{-e \sin \mathcal{E}_o}{2(1-e^2)}\;\frac{\partial e^2}{\partial a_j} + \frac{(1-e^2)^{1/2}}{1+e \cos \theta_o}\;\frac{\partial\,(e \sin \theta_o)}{\partial a_j} - \frac{e \sin \mathcal{E}_o}{1+e \cos \theta_o}\;\frac{\partial\,(e \cos \theta_o)}{\partial a_j}}$$

and

$$e \cos \xi_o = \frac{e^2 + e \cos \theta_o}{1 + e \cos \theta_o}$$

$$\frac{\partial (e \cos \xi_o)}{\partial a_j} = \frac{(1 + e \cos \theta_o)(\partial e^2/\partial a_j + \partial [e \cos \theta_o]/\partial a_j) - (e^2 + e \cos \theta_o)\partial (e \cos \theta_o)/\partial a_j}{(1 + e \cos \theta_o)^2}$$

$$\boxed{\frac{\partial (e \cos \xi_o)}{\partial a_j} = \frac{1}{1 + e \cos \theta_o} \frac{\partial e^2}{\partial a_j} + \frac{(1 - e^2)}{(1 + e \cos \theta_o)^2} \frac{\partial (e \cos \theta_o)}{\partial a_j}}$$

Also, since

$$e^2 = (e \cos \theta_o)^2 + (e \sin \theta_o)^2$$

it follows that

$$e \; de = e \cos \theta_o \; d(e \cos \theta_o) + e \sin \theta_o \; d(e \sin \theta_o)$$

so

$$\boxed{\frac{\partial e}{\partial a_j} = \cos \theta_o \frac{\partial (e \cos \theta_o)}{\partial a_j} + \sin \theta_o \frac{\partial (e \sin \theta_o)}{\partial a_j}}$$

Since

$$\xi_o = \tan^{-1} \left(\frac{e \sin \xi_o}{e \cos \xi_o} \right)$$

it follows that

$$\frac{\partial \xi_o}{\partial a_j} = \frac{1}{1 \div \tan^2 \xi_o} \quad \frac{e \cos \xi_o \dfrac{\partial (e \sin \xi_o)}{\partial a_j} - e \sin \xi_o \dfrac{\partial (e \cos \xi_o)}{\partial a_j}}{(e \cos \xi_o)^2}$$

so

$$\boxed{e \frac{\partial \xi_o}{\partial a_j} = \cos \xi_o \frac{\partial (e \sin \xi_o)}{\partial a_j} - \sin \xi_o \frac{\partial (e \cos \xi_o)}{\partial a_j}}$$

Since

$$M_o = \xi_o - e \sin \xi_o$$

it follows that:

$$\frac{\partial M_o}{\partial a_j} = \frac{\partial \xi_o}{\partial a_j} - e \cos \xi_o \frac{\partial \xi_o}{\partial a_j} - \sin \xi_o \frac{\partial e}{\partial a_j}$$

$$= (1 - e \cos \xi_o) \frac{\partial \xi_o}{\partial a_j} - \frac{\sin \xi_o}{2 e} \frac{\partial e^2}{\partial a_j}$$

$$\boxed{e \frac{\partial M_o}{\partial a_j} = (1 - e \cos \xi_o) \left(e \frac{\partial \xi_o}{\partial a_j} \right) - \frac{\sin \xi_o}{2} \frac{\partial e^2}{\partial a_j}}$$

At any time T after the initial observation, the time from perilune is

$$\boxed{t = t_o + T}$$

and the mean anomaly is

$$\boxed{M = nt}$$

The corresponding eccentric anomaly must be computed by solution of the equation

$$\boxed{M = \xi - e \sin \xi}$$

and the values of sin ξ and cos ξ found from ξ.

Then the trigonometric functions of the central angle are

$$\boxed{\sin\theta = \frac{(1-e^2)^{1/2}\sin\xi}{1-e\cos\xi}}$$

and

$$\boxed{\cos\theta = \frac{\cos\xi - e}{1-e\cos\xi}}$$

Now,

$$\tan\xi = \frac{\sin\xi}{\cos\xi} = \frac{(1-e^2)^{1/2}\sin\theta}{e+\cos\theta}$$

so

$$\frac{\partial\xi}{\partial a_j} = \frac{1-e\cos\xi}{(1-e^2)^{1/2}}\frac{\partial\theta}{\partial a_j} - \frac{\sin\xi}{2e(1-e^2)}\frac{\partial e^2}{\partial a_j}$$

Then from

$$M = \xi - e\sin\xi$$

it follows that

$$\frac{\partial M}{\partial a_j} = (1-e\cos\xi)\frac{\partial\xi}{\partial a_j} - \frac{\sin\xi}{2e}\frac{\partial e^2}{\partial a_j}$$

$$= \frac{(1-e\cos\xi)^2}{(1-e^2)^{1/2}}\frac{\partial\theta}{\partial a_j} - \frac{\sin\xi}{2e}\left(\frac{1-e\cos\xi}{1-e^2} + 1\right)\frac{\partial e^2}{\partial a_j}$$

$$e\frac{\partial M}{\partial a_j} = \frac{(1-e\cos\xi)^2}{(1-e^2)^{1/2}}\,e\,\frac{\partial\theta}{\partial a_j} - \frac{\sin\xi}{2}\frac{\partial e^2}{\partial a_j} - \frac{\sin\xi(1-e\cos\xi)}{2(1-e^2)}\frac{\partial e^2}{\partial a_j}$$

Now T is not varied when the orbit parameters are so $t - t_o$ should be held constant in taking partial derivatives with respect to the orbit parameters.

$$t - t_o = (M - M_o)/n$$

and

$$\frac{\partial (t - t_o)}{\partial a_j} = \frac{\partial \left[(M-M_o)/n\right]}{\partial a_j} = 0$$

so

$$\frac{1}{n} \frac{\partial (M-M_o)}{\partial a_j} - \frac{M-M_o}{n^2} \frac{\partial n}{\partial a_j} = 0$$

$$\frac{1}{n} \frac{\partial (M-M_o)}{\partial a_j} - \frac{M-M_o}{n^2} \frac{3n}{2E} \frac{\partial E}{\partial a_j} = 0$$

or

$$e \frac{\partial M_o}{\partial a_j} + e \frac{3 (M-M_o)}{2E} \frac{\partial E}{\partial a_j} = e \frac{\partial M}{\partial a_j}$$

Setting the two expressions for $e \frac{\partial M}{\partial a_j}$ equal to one another, we find

$$\boxed{e \frac{\partial \theta}{\partial a_j} = \frac{(1-e^2)^{1/2}}{(1-e \cos \xi)^2} \left\{ \frac{3(M-M_o)}{2E} e \frac{\partial E}{\partial a_j} + \left(e \frac{\partial M_o}{\partial a_j} \right) + \frac{\sin \xi}{2} \left(\frac{1 - e \cos \xi}{1-e^2} + 1 \right) \frac{\partial e^2}{\partial a_j} \right\}}$$

Setting ξ, θ, and M equal to ξ_o, θ_o, and M_o we find

$$e\,\frac{\partial(\theta-\theta_o)}{\partial a_j} = \frac{(1-e^2)^{1/2}}{(1-e\cos\xi)^2}\left\{\frac{3\,(M-M_o)}{2E}\;e\;\frac{\partial E}{\partial a_j}+\frac{\sin\xi}{2}\left(\frac{1-e\cos\xi}{1-e^2}+1\right)\frac{\partial e^2}{\partial a_j}\right\}$$

$$-\frac{(1-e^2)^{1/2}}{(1-e\cos\xi_o)^2}\left\{\frac{\sin\xi_o}{2}\left(\frac{1-e\cos\xi_o}{1-e^2}+1\right)\frac{\partial e^2}{\partial a_j}\right\}$$

$$+\,(1-e^2)^{1/2}\,e\,\frac{\partial M_o}{\partial a_j}\left\{\frac{1}{(1-e\cos\xi)^2}-\frac{1}{(1-e\cos\xi_o)^2}\right\}$$

$$=\frac{(1-e^2)^{1/2}e}{(1-e\cos\xi)^2}\left\{\frac{3\,(M-M_o)}{2E}\;\frac{\partial E}{\partial a_j}+\frac{\sin\xi}{2\,e}\left(\frac{1-e\cos\xi}{1-e^2}+1\right)\frac{\partial e^2}{\partial a_j}\right\}$$

$$-\frac{(1-e^2)^{1/2}e}{(1-e\cos\xi_o)^2}\left\{\frac{\sin\xi_o}{2\,e}\left(\frac{1-e\cos\xi_o}{1-e^2}+1\right)\frac{\partial e^2}{\partial a_j}\right\}$$

$$+\,(1-e^2)^{1/2}\,e\left(e\,\frac{\partial M_o}{\partial a_j}\right)\frac{-2\cos\xi_o+e\cos^2\xi_o+2\cos\xi-e\cos^2\xi}{(1-e\cos\xi)^2\,(1-e\cos\xi_o)^2}$$

$$\boxed{\frac{\partial(\theta-\theta_o)}{\partial a_j} = (1-e^2)^{1/2}\left\{ \frac{\frac{3(M-M_o)}{2E}\frac{\partial E}{\partial a_j} + \sin\mathcal{E}\left(\frac{1-e\cos\mathcal{E}}{1-e^2}+1\right)\frac{\partial e}{\partial a_j}}{(1-e\cos\mathcal{E})^2} - \frac{\sin\mathcal{E}_o\left(\frac{1-e\cos\mathcal{E}_o}{1-e^2}+1\right)\frac{\partial e}{\partial a_j}}{(1-e\cos\mathcal{E}_o)^2} + \frac{(\cos\mathcal{E}-\cos\mathcal{E}_o)\left(2-e\left[\cos\mathcal{E}+\cos\mathcal{E}_o\right]\right)}{(1-e\cos\mathcal{E})^2(1-e\cos\mathcal{E}_o)^2}\left(e\frac{\partial M_o}{\partial a_j}\right)\right.}$$

Now since

$$\boxed{r = \frac{H^2}{\mu(1+e\cos\theta)}}$$

and

$$\boxed{\dot{r} = \frac{\mu e}{H}\sin\theta}$$

it follows that

$$\boxed{\frac{\partial r}{\partial a_j} = 2\frac{r}{H}\frac{\partial H}{\partial a_j} - \frac{r}{1+e\cos\theta}\left[\cos\theta\frac{\partial e}{\partial a_j} - \sin\theta\left(e\frac{\partial\theta}{\partial a_j}\right)\right]}$$

and

$$\boxed{\frac{\partial\dot{r}}{\partial a_j} = \frac{\mu}{H}\left[\sin\theta\frac{\partial e}{\partial a_j} + \cos\theta\left(e\frac{\partial\theta}{\partial a_j}\right)\right] - \frac{\dot{r}}{H}\frac{\partial H}{\partial a_j}}$$

Now

$$\boxed{\begin{aligned} \sin(\theta - \theta_o) &= \sin\theta \cos\theta_o - \sin\theta_o \cos\theta_o \\ \cos(\theta - \theta_o) &= \cos\theta \cos\theta_o + \sin\theta \sin\theta_o \end{aligned}}$$

and the components of $\bar{r}$ in the x'y'z' co-ordinate system are:

$$\boxed{X' = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} r\cos(\theta - \theta_o) \\ r\sin(\theta - \theta_o) \\ 0 \end{bmatrix}}$$

and the components of $\dot{\bar{r}}$ in the same co-ordinates are

$$\boxed{\dot{X}' = \begin{bmatrix} \dot{x}' \\ \dot{y}' \\ \dot{z}' \end{bmatrix} = \begin{bmatrix} -\frac{H}{r}\sin(\theta - \theta_o) + \dot{r}\cos(\theta - \theta_o) \\ \frac{H}{r}\cos(\theta - \theta_o) + \dot{r}\sin(\theta - \theta_o) \\ 0 \end{bmatrix}}$$

The $\tilde{x}\,\tilde{y}\,\tilde{z}$ co-ordinate system is Earth-centered, non-rotating and right-handed with $\tilde{z}$ normal to the plane of the Moon's motion about the Earth and directed at an acute angle to the Earth's angular velocity vector. $\tilde{x}$ is in the direction of the Earth-Moon line at the time of the initial observation. Then the position and the velocity of the Moon with respect to the Earth are given by:

$$\boxed{\tilde{X}_m = \begin{bmatrix} \tilde{x}_m \\ \tilde{y}_m \\ \tilde{z}_m \end{bmatrix} = \begin{bmatrix} \rho_m \cos\omega_m T \\ \rho_m \sin\omega_m T \\ 0 \end{bmatrix}}$$

and

$$\boxed{\dot{\tilde{X}}_m = \begin{bmatrix} \dot{\tilde{x}}_m \\ \dot{\tilde{y}}_m \\ \dot{\tilde{z}}_m \end{bmatrix} = \begin{bmatrix} -\omega_m \tilde{y}_m \\ \omega_m \tilde{x}_m \\ 0 \end{bmatrix}}$$

in which ω_m is the angular rate of the Moon about the Earth and ρ_m is the Earth-Moon distance.

The xyz co-ordinate system is Earth-centered, right-handed and non-rotating. The z axis is in the direction of the Earth angular rotation vector and the x axis is in the plane of the prime meridian at the instant of the first observation. If λ is the observing station latitude (measured positive North) and α is the observing station longitude (measured positive East), then

$$\boxed{X_d = \begin{bmatrix} x_d \\ y_d \\ z_d \end{bmatrix} = \begin{bmatrix} \rho_e \cos\lambda \cos(\omega_e T + \alpha) \\ \rho_e \cos\lambda \sin(\omega_e T + \alpha) \\ \rho_e \sin\lambda \end{bmatrix}}$$

and

$$\boxed{\dot{X}_d = \begin{bmatrix} \dot{x}_d \\ \dot{y}_d \\ \dot{z}_d \end{bmatrix} = \begin{bmatrix} -\omega_e y_d \\ \omega_e x_d \\ 0 \end{bmatrix}}$$

in which ρ_e is the radius of the Earth and ω_e is the Earth's angular rate.

The latitude and longitude of the sub-lunar point on Earth at the instant of the first observation are L and ℓ respectively. The inclination between the plane of the orbit of the Moon about the Earth and the Earth's equator is β, as shown in Figure 1. It must be specified whether the angle γ in that figure is greater or less than $\pi/2$. Then

$$\boxed{\sin \gamma = \frac{\sin L}{\sin \beta}}$$

$$\boxed{\cos \gamma = \begin{cases} +\sqrt{1 - \sin^2 \gamma} & \text{if } \gamma \leq \pi/2 \\ -\sqrt{1 - \sin^2 \gamma} & \text{if } \gamma > \pi/2 \end{cases}}$$

$$\boxed{\sin (\phi - \ell) = \tan L \cot \beta}$$

$$\boxed{\cos (\phi - \ell) = \frac{\cos \gamma}{\cos L}}$$

$$\sin b = -\sin \beta \sin \left(\frac{\pi}{2} - \gamma \right)$$

$$\boxed{\sin b = -\sin \beta \cos \gamma}$$

$$\boxed{\cos b = +\sqrt{1 - \sin^2 b}}$$

$$\boxed{\sin \nu = \frac{-\tan b}{\tan \beta}}$$

$$\cos \nu = \frac{\cos\left(\frac{\pi}{2} - \gamma\right)}{\cos b}$$

$$\boxed{\cos \nu = \frac{\sin \gamma}{\cos b}}$$

$$\boxed{\begin{aligned} \cos \phi &= \cos(\phi - \ell)\cos \ell - \sin(\phi - \ell)\sin \ell \\ \sin \phi &= \sin(\phi - \ell)\cos \ell + \cos(\phi - \ell)\sin \ell \end{aligned}}$$

$$\boxed{\cos(\phi + \nu) = \cos \phi \cos \nu - \sin \phi \sin \nu}$$

$$\boxed{\sin(\phi + \nu) = \sin \phi \cos \nu + \sin \nu \cos \phi}$$

The rotation matrix K rotates vector components from the xyz system to the $\tilde{x}\tilde{y}\tilde{z}$ system.

$$K = \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix}$$

$$
\boxed{
\begin{aligned}
K_{11} &= \cos L \cos \ell \\
K_{12} &= \cos L \sin \ell \\
K_{13} &= \sin L \\
K_{21} &= \cos b \cos (\phi + \nu) \\
K_{22} &= \cos b \sin (\phi + \nu) \\
K_{23} &= \sin b \\
K_{31} &= K_{12} K_{23} - K_{13} K_{22} \\
K_{32} &= K_{13} K_{21} - K_{11} K_{23} \\
K_{33} &= K_{11} K_{22} - K_{12} K_{21}
\end{aligned}
}
$$

$$
\boxed{\tilde{X}_d = K X_d}
$$

$$
\boxed{\dot{\tilde{X}}_d = K \dot{X}_d}
$$

The rotation matrix L rotates vector components from the $\tilde{x}\ \tilde{y}\ \tilde{z}$ system to the $x'\ y'\ z'$ system. The relative angular positions of these two co-ordinate systems are shown in Figure 2.

$$
\cdot\, L = \begin{bmatrix} \cos\zeta & \sin\zeta & 0 \\ -\sin\zeta & \cos\zeta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\eta & \sin\eta \\ 0 & -\sin\eta & \cos\eta \end{bmatrix} \begin{bmatrix} \cos\xi & \sin\xi & 0 \\ -\sin\xi & \cos\xi & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

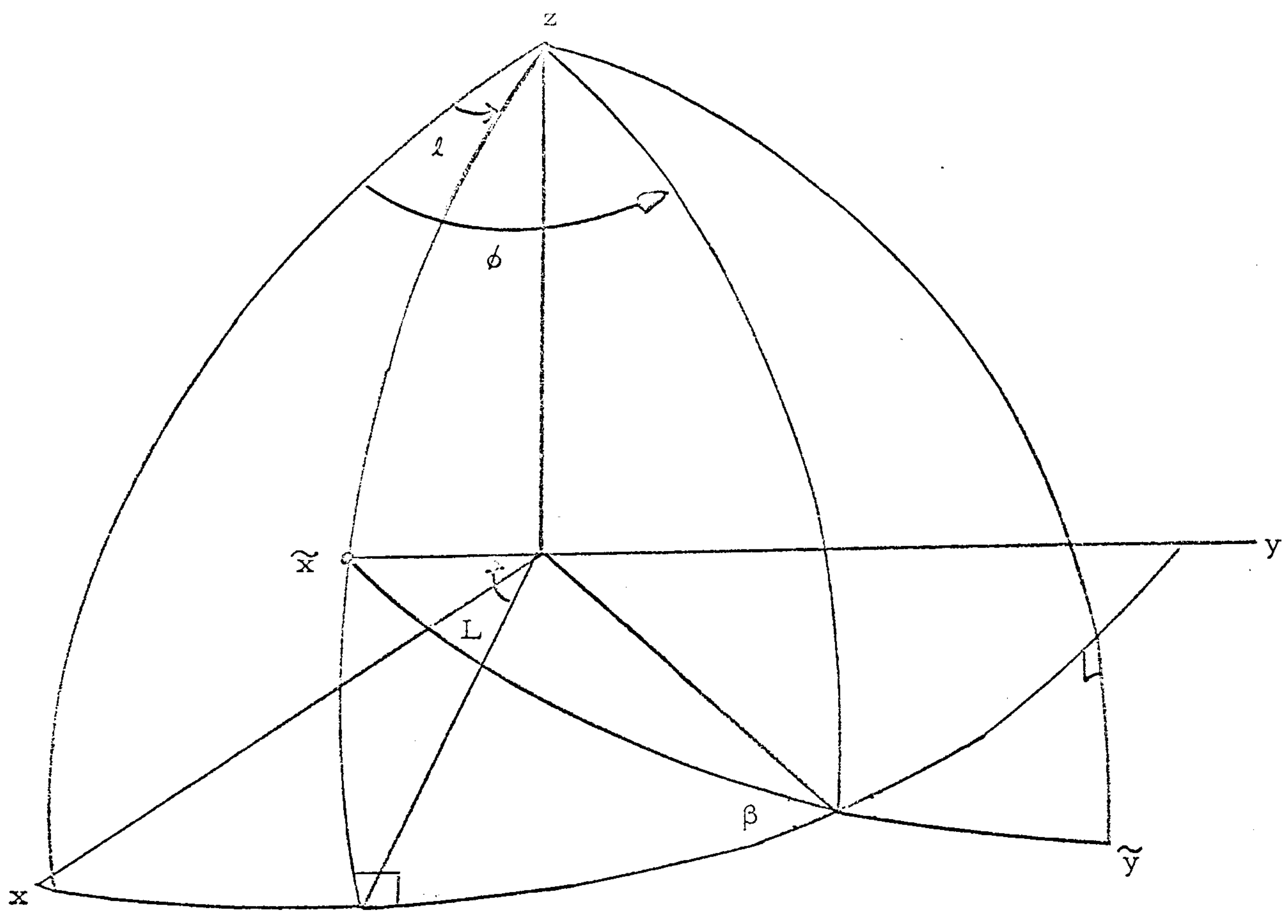
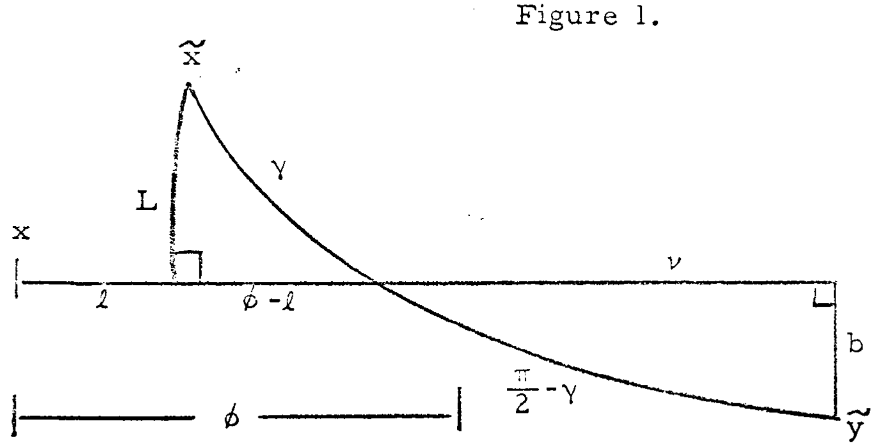


Figure 1.



Then the vectors $\bar{s}$ and $\dot{\bar{s}}$ are

$$\boxed{X'_s = \begin{bmatrix} x'_s \\ y'_s \\ z'_s \end{bmatrix} = X' + L\,(\widetilde{X}_m - \widetilde{X}_d)}$$

and

$$\boxed{\dot{X}'_s = \begin{bmatrix} \dot{x}'_s \\ \dot{y}'_s \\ \dot{z}'_s \end{bmatrix} = \dot{X}' + L\,(\dot{\widetilde{X}}_m - \dot{\widetilde{X}}_d)}$$

$$\boxed{s^2 = x'^2_s + y'^2_s + z'^2_s}$$

$$\boxed{\frac{\dot{s}}{s} = \frac{1}{s^2}\,(x'_s\,\dot{x}'_s + y'_s\,\dot{y}'_s + z'_s\,\dot{z}'_s)}$$

Let x^*, y^*, z^* be the co-ordinate system of a perturbed orbit. The components of the unit vectors $\hat{x}^*$, $\hat{y}^*$, $\hat{z}^*$ in the x', y', z' directions can be found as follows.

The perturbed initial position is on the $\hat{x}^*$ axis, so

$$\left[(a_1 + \Delta a_1)^2 + \Delta a_2^2 + \Delta a_3^2\right]^{\frac{1}{2}} \hat{x}^* = (a_1 + \Delta a_1)\,\hat{x}' + \Delta a_2\,\hat{y}' + \Delta a_3\,\hat{z}'$$

and, neglecting second and higher order terms,

$$\hat{x}^* = \hat{x}' + \frac{\Delta a_2}{a_1}\,\hat{y}' + \frac{\Delta a_3}{a_1}\,\hat{z}'$$

The perturbed angular momentum vector $\overline{H^*}$ is in the $\hat{z}^*$ direction.

$$\overline{H^*} = H^* \cdot \hat{z}^* = H^*_{x'} \hat{x}' + H^*_{y'} \hat{y}' + H^*_{z'} \hat{z}'$$

in which

$$H^*_{x'} = a_2^* a_6^* - a_3^* a_5^*$$

$$H^*_{y'} = a_3^* a_4^* - a_1^* a_6^*$$

$$H^*_{z'} = a_1^* a_5^* - a_2^* a_4^*$$

and

$$a_i^* = a_i + \Delta a_i$$

Then,

$$\hat{z}^* = \frac{H^*_{x'}}{H^*} \hat{x}' + \frac{H^*_{y'}}{H^*} \hat{y}' + \frac{H^*_{z'}}{H^*} \hat{z}'$$

Now

$$H^{*2} = (H^*_{x'})^2 + (H^*_{y'})^2 + (H^*_{z'})^2$$

so

$$H^* = H + \text{second order terms}$$

Thus,

$$\hat{z}^* = \frac{-a_5 \; \Delta a_3}{H} \hat{x}' + \frac{a_4 \; \Delta a_3 - a_1 \; \Delta a_6}{H} \hat{y}' + \hat{z}'$$

$\hat{y}^*$ is the cross product of $\hat{z}^*$ and $\hat{x}^*$, so

$$\hat{y}^* = \frac{-\Delta a_2}{a_1} \hat{x}' + \hat{y}' - \frac{a_4 \; \Delta a_3 - a_1 \; \Delta a_6}{H} \hat{z}'$$

Let $\overline{r^*}$ denote the perturbed value of $\overline{r}$ at any future time, and let

$$x^* = x' + X \qquad \text{in which } X = \sum_{i=1}^{6} \frac{\partial x^*}{\partial a_i} \Delta a_i$$

$$y^* = y' + Y \qquad \text{in which } Y = \sum_{i=1}^{6} \frac{\partial y^*}{\partial a_i} \Delta a_i$$

$$z^* = z' + Z \qquad \text{in which } Z = \sum_{i=1}^{6} \frac{\partial z^*}{\partial a_i} \Delta a_i$$

Next, let x'_p, y'_p, z'_p be the x', y', z' components of $\overline{r^*}$. Then,

$$x'_p = \overline{r^*} \bullet \hat{x}'$$

$$= x^* \hat{x}^* \bullet \hat{x}' + y^* \hat{y}^* \bullet \hat{x}' + z^* \hat{z}^* \bullet \hat{x}'$$

$$= x^* - y^* \frac{\Delta a_2}{a_1} - z^* \frac{\Delta a_3}{a_1}$$

and

$$\frac{x'_p - x'}{\Delta a_j} = \frac{X}{\Delta a_j} - \frac{y' + Y}{a_1} \frac{\Delta a_2}{\Delta a_j} - \frac{Z}{a_1} \frac{\Delta a_3}{\Delta a_j}$$

Letting Δa_j approach zero, $X/\Delta a_j$ becomes $\partial x'/\partial a_j$, and so on, so

$$\frac{\partial x'_p}{\partial a_j} = \frac{\partial x'}{\partial a_j} - \frac{y'}{a_1} \delta_{2j}$$

In like fashion

$$\frac{\partial y'_p}{\partial a_j} = \frac{\partial y'}{\partial a_j} + \frac{x'}{a_1} \delta_{2j}$$

$$\frac{\partial z'_p}{\partial a_j} = \frac{a_5 x' - a_4 y'}{H} \delta_{3j} + \frac{y'}{a_5} \delta_{6j}$$

and

$$\frac{\partial \dot{x}'_p}{\partial a_j} = \frac{\partial \dot{x}'}{\partial a_j} - \frac{\dot{y}'}{a_1} \delta_{2j}$$

$$\frac{\partial \dot{y}'_p}{\partial a_j} = \frac{\partial \dot{y}'}{\partial a_1} + \frac{\dot{x}'}{a_1} \delta_{2j}$$

$$\frac{\partial \dot{z}'_p}{\partial a_j} = \frac{a_5 \dot{x}' - a_4 \dot{y}_1}{H} \delta_{3j} + \frac{\dot{y}'}{a_5} \delta_{6j}$$

Then,

$$\boxed{R'_j = \begin{bmatrix} \dfrac{\partial x'_p}{\partial a_j} \\[2ex] \dfrac{\partial y'_p}{\partial a_j} \\[2ex] \dfrac{\partial z'_p}{\partial a_j} \end{bmatrix} = \begin{bmatrix} -y' \dfrac{\partial (\theta - \theta_o)}{\partial a_j} + \cos(\theta - \theta_o) \dfrac{\partial r}{\partial a_j} - \delta_{2j} \dfrac{y'}{a_1} \\[2ex] x' \dfrac{\partial (\theta - \theta_o)}{\partial a_j} + \sin(\theta - \theta_o) \dfrac{\partial r}{\partial a_j} + \delta_{2j} \dfrac{x'}{a_1} \\[2ex] \delta_{3j} \dfrac{a_5 x' - a_4 y'}{H} + \delta_{6j} \dfrac{y'}{a_5} \end{bmatrix}}$$

and

$$
\dot R'_j = \begin{bmatrix} \dfrac{\partial \dot x'_p}{\partial a_j} \\[2ex] \dfrac{\partial \dot y'_p}{\partial a_j} \\[2ex] \vdots \\[2ex] \dfrac{\partial \dot z'_p}{\partial a_j} \end{bmatrix} = \begin{bmatrix} -\left\{ \quad \right\}_1 \sin(\theta - \theta_o) + \left\{ \quad \right\}_2 \cos(\theta - \theta_o) - \delta_{2j} \dfrac{\dot y'}{a_1} \\[2ex] \left\{ \dfrac{1}{r} \dfrac{\partial H}{\partial a_j} - \dfrac{H}{r^2} \dfrac{\partial r}{\partial a_j} + \dot r \dfrac{\partial(\theta - \theta_o)}{\partial a_j} \right\}_1 \cos(\theta - \theta_o) + \delta_{2j} \dfrac{\dot x'}{a_1} \\[2ex] + \left\{ - \dfrac{H}{r} \dfrac{\partial(\theta - \theta_o)}{\partial a_j} + \dfrac{\partial \dot r}{\partial a_j} \right\}_2 \sin(\theta - \theta_o) \\[2ex] \delta_{3j} \dfrac{a_5 \dot x' - a_4 \dot y'}{H} + \delta_{6j} \dfrac{\dot y'}{a_5} \end{bmatrix}
$$

Finally,

$$
\frac{\partial \dot s}{\partial a_j} = \frac{1}{s} \left\{ X_s'^{\,T} \dot R'_j + \left[\dot X'_s - \frac{\dot s}{s} X'_s \right]^T R'_j \right\}
$$

and

$$
C_{i,j} = \sum_{p=1}^{N} \frac{\partial \dot s(T_p)}{\partial a_i} \frac{\partial \dot s(T_p)}{\partial a_j}
$$

in which T_p $(p = 1, \ldots, N)$ are the times of the N measurements, T_p being zero at the initial measurement.

The covariance matrix of the parameters is simply

$$
\left[\operatorname{cov} a_i a_j \right] = \sigma^2 \left[C_{i,j} \right]^{-1}
$$

in which σ^2 is the mean square Gaussian error in the measurements.

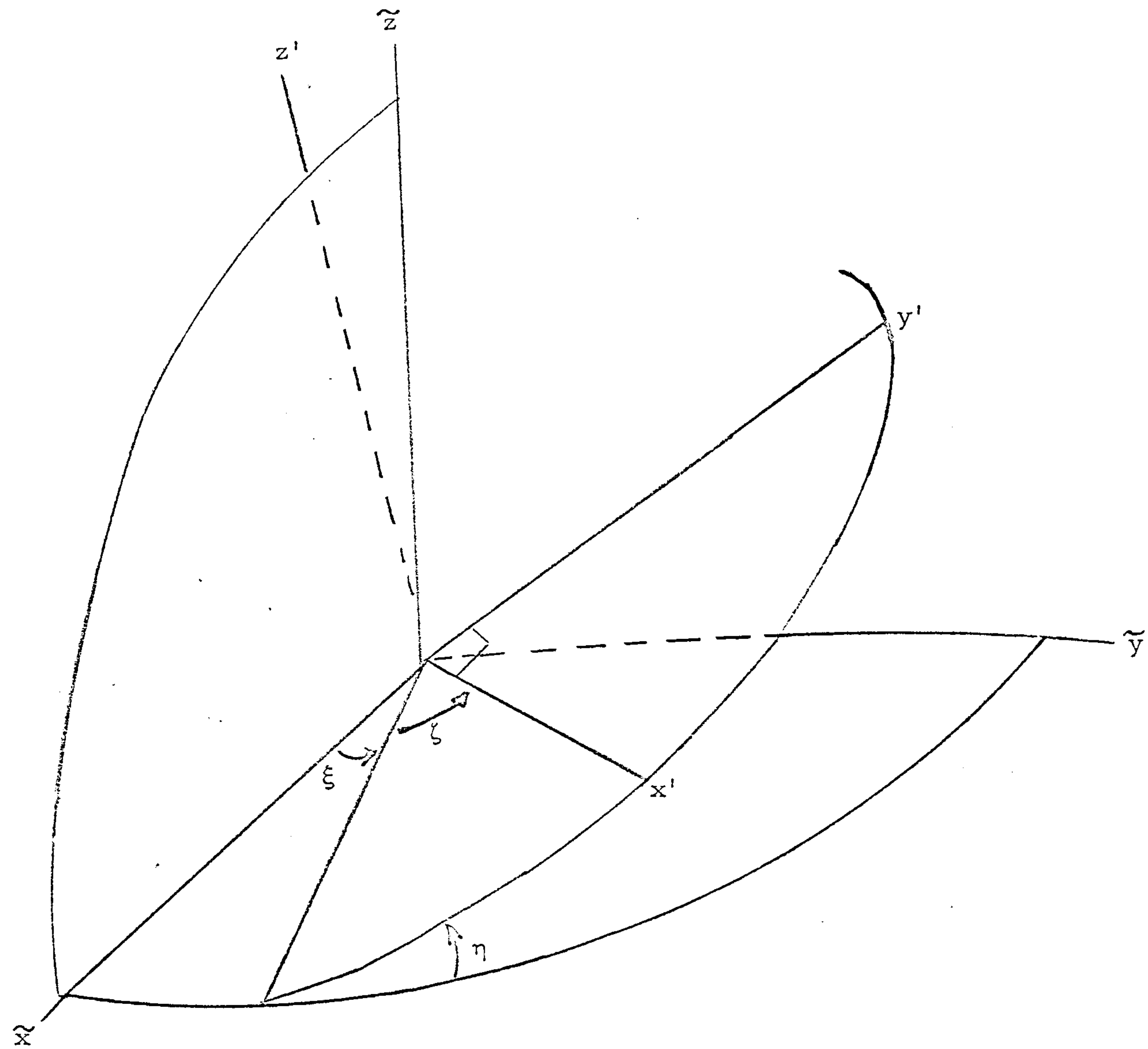


Figure 2.

APOLLO NOTE NO. 83

H. Engel
18 July 1963

USE OF RANGE AND RANGE RATE DATA

Range Only

From Apollo Note No. 43, if the measured quantities, f(t), have a fixed bias, b, and a zero mean stationary Gaussian noise, n(t), impressed, then

$$f_m(t) = f_c(t) + b + n(t)$$

and

$$\text{cov}\ (a_i, a_j) = \sigma^2\ (NC)^{-1} \qquad i, j = 1, \ldots, 7$$

in which σ^2 is the variance of the Gaussian noise,

$$f_c(t) = f_c\ (t,\ a_1, \ldots, a_M, b)$$

$$C_{i,j} = \sum_{k=1}^{N} \frac{\partial f_c\ (t_k)}{\partial a_i} \frac{\partial f_c\ (t_k)}{\partial a_j} \qquad i, j = 1, \ldots, 6$$

$$C_{7,j} = C_{j,7} = \sum_{k=1}^{N} \frac{\partial f_c\ (t_k)}{\partial a_j} \qquad j = 1, \ldots, 6$$

and

$$C_{7,7} = N$$

Now, if the quantity measured is the range from the DSIF or MSFN station to the vehicle orbiting the Moon, then from Apollo Note No. 82,

$$\bar{s} = \bar{X}_m - \bar{X}_d + \bar{r}$$

$$s^2 = \bar{s} \cdot \bar{s}$$

$$\bar{s}\,\frac{\partial s}{\partial s_j} = \bar{s} \cdot \frac{\partial \bar{s}}{\partial a_j}$$

$$= \bar{s} \cdot \frac{\partial \bar{r}}{\partial a_j}$$

or

$$\frac{\partial f_c}{\partial a_j} \triangleq \frac{\partial s}{\partial a_j} = \frac{1}{s} \quad \bar{s} \cdot \frac{\partial \bar{r}}{\partial a_j} .$$

Now, expressions for s, $\bar{s}$, and $\frac{\partial \bar{r}}{\partial a_i}$ are available from Apollo Note No. 82, so it is simple (in theory) to determine the covariance matrix of orbit parameters and radar range bias using this note and Apollo Note No. 82.

Range and Range Rate

From Apollo Notes No. 43 and 82, the likelihood function of the data is:

$$L = \frac{1}{\left(2\pi\right)^{\frac{N_1+N_2}{2}} \sigma_R^{N_1} \quad \sigma_{\dot{R}}^{N_2}} \exp\left(-\mathcal{L}\right)$$

in which N_1 is the number of range measurements, N_2 the number of range rate measurements, and

$$\mathcal{L} = \frac{1}{2\sigma_R^2} \sum_{k=1}^{N_1} \left[R_m(k) - R_c(a_i, k) - b \right]^2$$

$$+ \frac{1}{2\sigma_{\dot{R}}^2} \sum_{\ell=1}^{N_2} \left[\dot{R}_m(\ell) - \dot{R}_c(a_i, \ell) \right]^2$$

To obtain the covariance of the most likely a_i's and b, set $\partial \mathcal{L}/\partial a_i$ and $\partial \mathcal{L}/\partial b$ equal to zero, and, assuming N_1 and N_2 are sufficiently great, obtain a set of seven equations linear in the random perturbation quantities Δa_i and Δb.

This leads to:

$$C_{i,j} = \frac{1}{\sigma_R^2} \sum_{k=1}^{N_1} \frac{\partial R_c(k)}{\partial a_i} \frac{\partial R_c(k)}{\partial a_j} + \frac{1}{\sigma_{\dot{R}}^2} \sum_{\ell=1}^{N_2} \frac{\partial \dot{R}_c(\ell)}{\partial a_i} \frac{\partial \dot{R}_c(\ell)}{\partial a_j} \quad ; i, j = 1, \ldots, 6$$

$$C_{7,j} = C_{j,7} = \frac{1}{\sigma_R^2} \sum_{k=1}^{N_1} \frac{\partial R_c(k)}{\partial a_j}$$

$$C_{7,7} = \frac{N_1}{\sigma_R^2}$$

and

$$\text{cov}\,(a_i, a_j) = C^{-1} \qquad i, j = 1, \ldots, 7$$

APOLLO NOTE NO. 85

J. Holdsworth
22 July 1963

THE EQUIVALENCE OF DATA PROCESSING SCHEMES IN A LINEARIZED ERROR ANALYSIS

The purpose of this note is to demonstrate rigorously the statistical equivalence of certain pairs of data processing schemes under the assumption of a linear propagation of random errors in the estimators of certain orbital parameters.

First, we consider that we are observing a time series $\dot{R}_m(t)$ which may be represented as:

$$\dot{R}_m(t) = \dot{R}_c(a_1, \ldots a_6, t) + n(t) \tag{1}$$

where n(t) is a white stationary zero mean gaussian process with variance σ_n^2. We assume that we know the functional dependence of $\dot{R}_c(a_1 \ldots a_6, t)$ on the orbit parameters a_i. Then, on the basis of N_1 pieces of data $\dot{R}_m(1), \ldots \dot{R}_m(N_1)$, where $N_1 \geq 6$, we compute the maximum likelihood estimates $\hat{a}_1, \ldots \hat{a}_6$ of the parameters $a_1, \ldots a_6$.

Apollo Note No. 43 gives a method for computing the covariance matrix of the errors in the estimators of the parameters as a function of the noise variance σ_n^2 and the number of independent observations N_1. It will be recalled that the analysis in that note was based upon the assumption that the number of pieces of data N_1 was sufficiently large so that the following linearization was valid:

$$\dot{R}_c(\hat{a}_1, \ldots \hat{a}_6, t) = \dot{R}_c(a_1, \ldots a_6, t) + \sum_{i=1}^{6} \frac{\partial \dot{R}_c}{\partial a_i}(a_1, \ldots a_6, t)\, \Delta\hat{a}_i \tag{2}$$

where

$$\Delta\hat{a}_i = \hat{a}_i - a_i.$$

Let $\Delta\hat{a}$ denote the column estimator error vector whose components are: $\hat{a}_i - a_i$ for $i = 1, 2, \ldots 6$. It was shown in Apollo Note No. 43 that the covariance matrix of the estimator errors is given by:

$$\mathrm{Cov}\left(\Delta\hat{a}\,(N_1)\right) = \sigma_n^2\, C^{-1}\,(N_1) \tag{3}$$

where the i, jth element of the matrix C is given by:

$$C_{i,j}(N_1) = \sum_{k=1}^{N_1} \frac{\partial \dot{R}_c}{\partial a_i}(a_1, \ldots a_6, t_k)\, \frac{\partial \dot{R}_c}{\partial a_j}(a_1, \ldots a_6, t_k) \tag{4}$$

Furthermore, if the parameters were estimated on the basis of $N_i + N_2$ observations we would have

$$\mathrm{Cov}\left(\Delta\hat{a}\,(N_1+N_2)\right) = \sigma_n^2\, C^{-1}\,(N_1+N_2) \tag{5}$$

where:

$$C_{i,j}\,(N_1+N_2) = \sum_{k=1}^{N_1+N_2} \frac{\partial \dot{R}_c}{\partial a_i}(a_1, \ldots a_6, t_k)\, \frac{\partial \dot{R}_c}{\partial a_j}(a_1, \ldots a_6, t_k)$$

$$= \sum_{k=1}^{N_1} \frac{\partial \dot{R}_c}{\partial a_i}(a_1, \ldots a_6, t_k)\, \frac{\partial \dot{R}_c}{\partial a_j}(a_1, \ldots a_6, t_k)$$

$$+ \sum_{k=N_1+1}^{N_2} \frac{\partial \dot{R}_c}{\partial a_i}(a_1, \ldots, a_6, t_k)\, \frac{\partial \dot{R}_c}{\partial a_j}(a_1, \ldots a_6, t_k). \tag{6}$$

In Equations (3) -(6), we have made the dependence of the covariance matrices upon the smoothing times notationally explicit by writing Cov $\left(\Delta \hat{a}\ (N_1)\right)$.. etc.

Now assume that N_1 observations are made first and that an estimator vector $\hat{a}\ (N_1)$ is computed from the observations $\dot{R}_m\ (t_1)$.... $\dot{R}_m(t_{N_1})$, where the ith component of the estimator vector $\hat{a}\ (N_1)$ is the maximum likelihood estimate of a_i. Then it follows from what has been said that the covariance matrix of the estimator vector $\hat{a}\ (N_1)$ is given by the expression in (3).

Next assume that the same parameters are to be estimated from N_2 observations $\dot{R}_m\ (t_{N_1}+1)$ ...., $\dot{R}_m(t_{N_1+N_2})$ without the knowledge of $\dot{R}_m\ (t_1)$.... $\dot{R}_m\ (t_{N_1})$. Denote the resulting estimator vector based upon $\dot{R}_m\ (t_{N_1+1})$... $\dot{R}_m\ (t_{N_1+N_2})$ by $\hat{a}\ (N_2)$. Then if N_2 is sufficiently large so that the linearization in (2) holds, we may write:

$$\text{Cov}\left(\Delta \hat{a}\ (N_2)\right) = \sigma_n^2\ C^{-1}\ (N_2) \tag{7}$$

where:

$$C_{i,j}\ (N_2) = \sum_{k=N_1+1}^{N_1+N_2} \frac{\partial \dot{R}_c}{\partial a_i}\ (t_k)\ \frac{\partial \dot{R}_c}{\partial a_j}\ (t_k) \tag{8}$$

It is important to note that the estimator vector $\hat{a}\ (N_2)$ was computed without assuming that the computed value of the estimator vector $\hat{a}\ (N_1)$ was known and conversely. What we wish to show is that knowing only the computed estimator vectors $\hat{a}\ (N_1)$ and $\hat{a}\ (N_2)$ allows us to form a new estimator vector of the parameters such that the resultant accuracy is the same as if we were able to make the total number, N_1+N_2, of observations $\dot{R}_m(t_1)$... $\dot{R}_m(t_{N_1})$, $\dot{R}_m\ (t_{N_1+1})$... $\dot{R}_m(t_{N_1+N_2})$ first and then estimate the parameters.

To show this, we assume that we have computed $\hat{a}\ (N_1)$ and $\hat{a}\ (N_2)$ as described above. Knowing the estimator vectors $\hat{a}\ (N_1)$ and

$\hat{a}(N_2)$, we define the new estimator vector $\tilde{a}(N_1, N_2)$ by:

$$\tilde{a}(N_1, N_2) = \mathrm{Cov}\left(\Delta\hat{a}(N_1+N_2)\right)\left[\mathrm{Cov}^{-1}\left(\Delta\hat{a}(N_1)\right)\hat{a}(N_1) + \mathrm{Cov}^{-1}\left(\Delta\hat{a}(N_2)\right)\hat{a}(N_2)\right]. \tag{9}$$

Note that the new estimator $\tilde{a}(N_1, N_2)$ is obtained from linear operations on the vectors $\hat{a}(N_1)$ and $\hat{a}(N_2)$, and that $\tilde{a}(N_1, N_2)$ depends upon the observed data only through the estimator vectors $\hat{a}(N_1)$ and $\hat{a}(N_2)$. The proof of our assertion will consist of showing that:

$$\mathrm{Cov}\left(\Delta\tilde{a}(N_1, N_2)\right) = \mathrm{Cov}\left(\Delta\hat{a}(N_1+N_2)\right) \tag{10}$$

If we let a denote the column vector whose ith component is a_i, then subtracting a from both sides of (9) yields the following expression for the estimator error vector $\Delta\tilde{a}(N_1, N_2)$.

$$\Delta\tilde{a}(N_1, N_2) = \mathrm{Cov}\left(\Delta\hat{a}(N_1+N_2)\right)\left[\mathrm{Cov}^{-1}\left(\Delta\hat{a}(N_1)\right)\Delta\hat{a}(N_1) + \mathrm{Cov}^{-1}\left(\Delta\hat{a}(N_2)\right)\Delta\hat{a}(N_2)\right] \tag{11}$$

The covariance matrix of the estimator errors $\Delta\tilde{a}(N_1, N_2)$ is given by:

$$\mathrm{Cov}\left(\Delta\tilde{a}(N_1, N_2)\right) = E\left[\Delta\tilde{a}(N_1, N_2)\left(\Delta\tilde{a}(N_1, N_2)\right)^T\right] \tag{12}$$

where T denotes matrix transposition and E is the expected value operator which is integration over the observation space with respect to the joint distribution of the components of $\Delta\hat{a}(N_1)$ and $\Delta\hat{a}(N_2)$. Using the symmetry of the matrix $\mathrm{Cov}\left(\Delta\hat{a}(N_1+N_2)\right)$ we may explicitly write:

$$\mathrm{Cov}\left(\Delta\tilde{a}(N_1, N_2)\right) = \mathrm{Cov}\left(\Delta\hat{a}(N_1+N_2)\right)\left[\mathrm{Cov}^{-1}\left(\Delta\hat{a}(N_1)\right) + \mathrm{Cov}^{-1}\left(\Delta\hat{a}(N_2)\right)\right]\mathrm{Cov}\left(\Delta\hat{a}(N_1+N_2)\right) \tag{13}$$

where in (13) we have used the fact that the maximum likelihood error vectors $\Delta\hat{a}(N_1)$, $\Delta\hat{a}(N_2)$ are independently distributed with asymptotically multivariate normal distribution with zero means so that terms of the form:

$$E\left\{\mathrm{Cov}^{-1}\left(\Delta\hat{a}(N_1)\right)\Delta\hat{a}(N_1)\left(\Delta\hat{a}(N_2)\right)^T\mathrm{Cov}^{-1}\left(\Delta\hat{a}(N_2)\right)\right\} = 0 \tag{14}$$

thus yielding (13).

However, from Equations (4), (6), and (8), we see that:

$$\mathrm{Cov}^{-1}\left[\Delta\hat{a}(N_1+N_2)\right] = \mathrm{Cov}^{-1}\left[\Delta\hat{a}(N_1)\right] + \mathrm{Cov}^{-1}\left[\Delta\hat{a}(N_2)\right] \tag{15}$$

so that

$$\mathrm{Cov}\left(\Delta\tilde{a}(N_1,N_2)\right) = \mathrm{Cov}\left(\Delta\hat{a}(N_1+N_2)\right) \tag{16}$$

which proves our assertion.

Intuitively we may say that a priori distributional knowledge obtained from prior observations has a modifying effect on our later data so that the effective sample size is increased. It is worthwhile to note that if the noise is auto-correlated, then quantities such as the expression given by (14) will not in general vanish and our assertion will no longer be true.

Next we consider the following situation. We assume that observations are made as before over a given interval of time -- say $\dot{R}_m(t_1), \ldots \dot{R}_m(t_N)$. On the basis of these observations we wish to estimate the values of the components of the position and velocity vectors at some initial time t = 0. Let these quantities be denoted by $x_i(0)$ where i = 1, . . . 6.

The value of the quantities $x_i(t)$ at an arbitrary time t depends upon the quantities $x_i(0) \ldots x_i(0)$ and conversely. Thus there exist relations of the form:

$$x_i(t) = f_i\left(x_1(0), \ldots x_6(0)_i t\right) \qquad i=1,\ldots 6 \tag{17}$$

and

$$x_i(0) = g_i\left(x_1(t), \ldots x_i(t)\right)$$

Having originally estimated the quantities $x_i(0)$ from our observations we may use (17) to predict values for the $x_i(t)$ based upon the estimators $\hat{x}_1(0) \ldots \hat{x}_2(0)$. Because of errors in the estimator state vector $\hat{x}(0)$, there will also be errors in the predicted quantities $\tilde{x}_1(t) \ldots \tilde{x}_2(t)$. Since:

$$\Delta\tilde{x}_i(t) = \sum_{k=1}^{6} \frac{\partial f_i}{\partial x_n(0)} \Delta x_n(0) \tag{18}$$

Then:

$$\Delta\tilde{x}_i(t)\,\Delta\tilde{x}_j(t) = \sum_{\ell=1}^{6} \sum_{k=1}^{6} \frac{\partial f_i}{\partial x_n(0)} \frac{\partial f_j}{\partial x_\ell(0)} \Delta x_k(0)\,\Delta x_\ell(0). \tag{19}$$

From (19) we see that the predicted covariance matrix for the quantities $\tilde{x}_i(t)$ may be written

$$\mathrm{Cov}\left(\Delta\tilde{x}(t)\right) = J\ \mathrm{Cov}\left(\Delta\hat{x}(0)\right) J^T \tag{20}$$

where J is the Jacobian matrix whose i, jth element is $\dfrac{\partial f_i}{\partial x_j(0)}$.

On the other hand, instead of first estimating $x_i(0)$ from the data and then predicting ahead to the $\tilde{x}_i(t)$ on the basis of the estimators $\hat{x}_i(0)$, we could have estimated the $x_i(t)$ quantities directly from the data. Let these direct estimates be denoted by $\hat{x}_i(t)$. We wish to show that under the assumptions of a linear propagation of errors that the two methods are equivalent, i.e., that

$$\mathrm{Cov}\left(\Delta\hat{x}(t)\right) = \mathrm{Cov}\left(\Delta\tilde{x}(t)\right) \tag{21}$$

To do this we recall that the i, jth element of the inverse of the matrix $\mathrm{Cov}\,(\Delta\hat{x}(t)$ is given by:

$$\hat{C}_{i,j} = \frac{1}{\sigma_n^2} \sum_{k=1}^{N} \frac{\partial \dot{R}_c(t_k)}{\partial x_i(t)} \frac{\partial \dot{R}_c(t_k)}{\partial x_j(t)} \tag{22}$$

On the other hand, from Equation (20), the inverse of $\mathrm{Cov}\left(\Delta\tilde{x}(t)\right)$ may be written:

$$\mathrm{Cov}^{-1}\left(\Delta \tilde{x}(t)\right) = (J^T)^{-1}\ \mathrm{Cov}^{-1}\left(\Delta \hat{x}(0)\right) J^{-1} \tag{23}$$

Since the inverse of the Jacobian matrix is the matrix whose i, jth element is $\frac{\partial g_i}{\partial x_j(t)}$ and since the i, jth element of $\mathrm{Cov}^{-1}\left(\Delta x(0)\right)$ is

$$\frac{1}{\sigma^2} \sum_{k=1}^{N} \frac{\partial \dot{R}_c(t_k)}{\partial x_i(0)} \frac{\partial \dot{R}_c(t_k)}{\partial x_j(0)}$$

then it follows that the i, jth element of $\mathrm{Cov}^{-1}\left(\Delta \tilde{x}(t)\right)$ may be written:

$$\tilde{C}_{i,j} = \frac{1}{\sigma^2} \sum_{\ell=1}^{6} \sum_{m=1}^{6} \sum_{n=1}^{N} \frac{\partial g_\ell}{\partial x_i(t)} \frac{\partial \dot{R}_c(t_n)}{\partial x_\ell(0)} \frac{\partial \dot{R}_c(t_n)}{\partial x_m(0)} \frac{\partial g_m}{\partial x_j(t)} \tag{24}$$

However, for fixed k, we note that

$$\sum_{\ell=1}^{6} \frac{\partial g_\ell}{\partial x_i(t)} \frac{\partial \dot{R}_c(t_n)}{\partial x_\ell(0)} = \frac{\partial \dot{R}_c(t_n)}{\partial x_i(t)}$$

and (25)

$$\sum_{m=1}^{6} \frac{\partial g_m}{\partial x_i(t)} \frac{\partial \dot{R}_c(t_n)}{\partial x_m(0)} = \frac{\partial \dot{R}_c(t_n)}{\partial x_j(t)}$$

thus: $\hat{C}_{i,j} = \tilde{C}_{i,j}$ and

$$\mathrm{Cov}\left(\Delta \tilde{x}(t)\right) = \mathrm{Cov}\left(\Delta \hat{x}(t)\right) \tag{26}$$

which completes the proof of equivalence.

APOLLO NOTE NO. 93

H. Engel
14 August 1963

GROUND SYSTEM COMPUTATION OF LEM ORBIT

In the various schemes for ground support system computation of the LEM orbit on the basis of DSIF or MSFN radar data, there is the ever-present question of whether the computations can be performed rapidly enough to be of use. This note is intended to provide an answer to the question.

The conventional technique employed on deep-space shots is to acquire data for a long time, and then find the mean square error between the observations and the values of the quantity observed calculated on the basis of assumed or previously determined values of the trajectory parameters. The parameters are then adjusted and the computations repeated. The process continues until the mean square error is sufficiently small or can be reduced no further. The computation flow diagram is as follows for one kind of measurement only, say, Doppler velocity,

Given a priori values of orbit parameters $a_1, \ldots, a_s$

Obtain data measurements $M_1, M_2, \ldots, M_N$

Using a priori values of parameters

Compute expected values $E_1, E_2, \ldots, E_N$ of measurements, and $\partial E_1/\partial a_j, \ldots, \partial E_N/\partial a_j$ by integrating the differential equations of the orbit.

(1) ⟶ Compute $\sigma^2 = \frac{1}{N} \sum_{i=1}^{N} (M_i - E_i)^2$,

$$\frac{\partial \sigma^2}{\partial a_j} = \frac{2}{N} \sum_{i=1}^{N} \left(M_i - E_i - \sum_{k=1}^{s} \Delta a_k \frac{\partial E}{\partial a_k} \right) \frac{\partial E}{\partial a_j}$$

Is σ^2 small enough or as small as possible ?

yes → End

no ↓

On the basis of σ^2 and $\frac{\partial \sigma^2}{\partial a_j}$ modify $a_1, \ldots, a_s$

Using these parameters compute E_i and $\partial E_i / \partial a_j$ $(i=1, \ldots, N)$ by integrating the differential equations of the orbit.

Go to (1) .

In this method of computation the computations up to the first yes-no decision can be performed concurrently with the receipt of data, but the computations necessary for the remaining computations can not be instituted until all the data is available. As a result in the case of deep-space shots, there is a long delay between receipt of the last data point and determination of the orbit parameters. According to JPL, the time required to perform an iteration is currently 20 to 25 minutes, and might possibly be reduced to 10 minutes by reprogramming. Two or three iterations have been found necessary. Further, the number of parameters considered by JPL is far more than 6, since the speed of light, the mass of the Moon and other quantities are considered as parameters. Still further, the number of observations used in the deep-space shots is far greater than will be used in ground support of the LEM.

Assuming that the gravitational field of the Moon and the ephemeris of the Moon have been determined to sufficient accuracy from previous lunar orbiting shots, and assuming that from burn-out to passing out of sight behind the Moon takes only 30 minutes, so that only 30 one-minute observations can be made, the time required to perform the necessary calculations for orbit parameter determination should be reduced by at least an order of magnitude. Use of the IBM 7094 instead of the IBM 7090 will reduce the time by almost another order of magnitude because of built-in double precision operations.

Since 30 observations is not a very large number of observations, it is not desirable to gain additional time for computation by decreasing the number of observations. Other means for reducing the time required for computation after receipt of the last data point can be found. Two approaches to this problem are possible.

The first approach starts the iteration procedure at, say, the 27th observation. The starting point for the second iteration, which uses 28 data points, is the orbit parameters determined in this fashion. In like manner the third iteration uses 29 observations, and the fourth 30 iterations, so the computations are completed with just one iteration after the last observation.

The second approach uses the Kalman-Schmidt method of parameter determination. This method is described in Apollo Note No. 88. The Kalman-Schmidt method employs a nominal or reference trajectory that has been pre-calculated in order to linearize the problem and to reduce the amount of real-time computation. If the actual trajectory is "close enough" to the reference trajectory, the Kalman-Schmidt method can be employed to produce orbit parameters within milliseconds after the final observation.

Using the lengthier procedure, the estimated time to find improved orbit parameters after the last observation is of the order of 3 seconds using range rate from one station with no biases, or 7 seconds using range, range rate and angle from three stations including biases in range and angle.

These computation times will be increased substantially if triple precision operations are required instead of double precision operations.

The computation times will increase slightly if more complete expressions for the gravitational field of the Moon must be employed.

The time to compute guidance instructions is negligible compared to the time required for orbit determination.

APPENDIX A

Encke's method is suitable for orbit calculations when near enough to one body that the effects of the gravitational fields of other bodies can be considered as small perturbations. This is the case for the LEM ascent and rendezvous trajectories.

Let $\overline{R} = \overline{R}(t)$ be the orbit relative to the Moon resulting from initial conditions of position and velocity, considering only the radially symmetric, inverse square component of the Moon's gravitational field.

$$\ddot{\overline{R}} = -\frac{\mu_M}{R^3}\overline{R}$$

in which μ_M is the gravitational constant of the Moon.

Let $\overline{r} = \overline{r}(t)$ be the actual motion relative to the Moon, and let

$$\overline{\rho} \triangleq \overline{r} - \overline{R}$$

Then

$$\ddot{\overline{r}} = -\frac{\mu_M}{r^3}\overline{r} + \overline{P}$$

and

$$\ddot{\overline{\rho}} = -\mu_M\left(\frac{\overline{r}}{r^3} - \frac{\overline{R}}{R^3}\right) + \overline{P}$$

in which $\overline{P}$ is the perturbing acceleration.

Letting

$$Q \triangleq \frac{1}{2}\left[\left(\frac{r}{R}\right)^2 - 1\right]$$

it is found that

$$Q = \frac{\overline{\rho}\cdot(\overline{R} + \overline{\rho}/2)}{R^2}$$

and

$$\ddot{\overline{\rho}} = \frac{-\mu_M}{R^3}\left[\overline{\rho} - \overline{r}\;F(Q)\right] + \overline{P}$$

where

$$F(Q) = 1 - (R/r)^3 = 1 - (1 + 2Q)^{-\frac{3}{2}}$$

$$= \sum_{j=1}^{\infty} C_j Q^j$$

For the orbits of interest, Q is a small quantity, R being of the order of 1.75×10^6 m and r differing from it by at most of the order of 10^4 m, so that $|Q| < 3 \times 10^{-5}$.

Further, the first few terms in the expansion for F (Q) are given by

$$F(Q) = 3Q - \frac{15}{2} Q^2 + \frac{35}{2} Q^3 - \frac{315}{8} Q^4 + \ldots$$

and since Q is of the order of 3×10^{-5} at most, it follows that the ratio of the third term to the first term in the above expression is between 0 and 5.25×10^{-9}. Still further the ratio of r F (Q) to ρ is

$$\frac{r\,F(Q)}{\rho} \doteq \frac{r\,3\,Q}{\rho} \doteq 3\,\frac{r\left(\frac{r - R}{R}\right)^2}{\rho}$$

$$< \frac{r}{\rho}\left(\frac{\rho}{R}\right)^2$$

$$< \frac{\rho}{R}$$

$$< 5.7 \times 10^{-3}$$

hence

$$\frac{r\,\frac{35}{2}\,Q^3}{\rho} < 5.7 \times 10^{-3} \times 5.25 \times 10^{-9}$$

$$< 3 \times 10^{-11}$$

Thus, the third and succeeding terms in F (Q) may be neglected, and

$$F(Q) \doteq 3Q(1 - \frac{5}{2}Q)$$

The perturbing acceleration $\overline{P}$ may be written as

$$\overline{P} = \overline{P_1} + \overline{P_2} + \overline{P_3} + \overline{P_4}$$

in which

$\overline{P_1}$ = results from the remaining portion of the Moon's gravitational field.

$\overline{P_2}$ = results from the Earth's gravitational field.

$\overline{P_3}$ = results from the Sun's radiation pressure.

$\overline{P_4}$ = results from the Sun's gravitational field.

The gravitational potential of the Moon, in excess of its radially symmetric, inverse square portion, may be expressed as

$$U_M = \frac{G}{2r^3} (A_1 + A_2 + A_3 - 3 I)$$

in which G is the universal gravitational constant, A_i is the moment of inertia about the principal axis u_i, and

$$I = \frac{1}{r^2} (A_1 u_1^2 + A_2 u_2^2 + A_3 u_3^2)$$

The corresponding acceleration has components

$$P_{1,j} = -\frac{\partial U}{\partial u_j} = \frac{3G}{2R^4} \frac{u_j}{R} (A_1 + A_2 + A_3 - 5I + 2 A_j)$$

$$j = 1, 2, 3$$

The perturbing effect of the Earth is due to the difference in acceleration of the Moon and vehicle resulting from the Earth's gravitational field. Letting $\overline{R}_{EM}$ be the vector from the Earth to the Moon, then for the radially symmetric, inverse square portion of the field,

$$\overline{P_2} = \frac{\mu_E \, (\overline{R}_{EM} + \overline{r})}{\left| \overline{R}_{EM} + \overline{r} \right|^3} + \frac{\mu_E \, \overline{R}_{EM}}{R_{EM}^3}$$

$$= \frac{\mu_E}{R_{EM}^2} \left[\frac{\overline{R}_{EM} + \overline{r}}{R_{EM}} \left(1 + 2 \, \frac{\overline{R}_{EM} \cdot \overline{r}}{R_{EM}^2} - \frac{r^2}{R_{EM}^2} \right)^{-\frac{3}{2}} - \frac{\overline{R}_{EM}}{R_{EM}} \right]$$

Observe that r/R_{EM} is of the order of 5×10^{-3} or less. Letting

$$\alpha \triangleq \frac{2 \, \overline{R}_{EM} \cdot \overline{r}}{R_{EM}^2} + \frac{r^2}{R_{EM}^2}$$

it follows that

$$\left| \alpha \right| < 2 \, (5 \times 10^{-3})$$

$$< 10^{-2}$$

Further, μ_E / R^2_{EM} is of the order of 3×10^{-6} m/sec ^{2}. Then

$$\overline{P_2} = \frac{-\mu_E}{R_{EM}^2} \left[\frac{\overline{r}}{R_{EM}} + \frac{\overline{R}_{EM} + \overline{r}}{R_{EM}} \left(- \frac{3}{2} \alpha + \frac{15}{8} \alpha^2 - \frac{35}{16} \alpha^3 + \frac{315}{128} \alpha_4 - \ldots \right) \right]$$

$$\doteq - 3 \times 10^{-6} \left[3 \times 10^{-5} + 1 \left(- \frac{3}{2} 10^{-2} + \frac{15}{8} 10^{-4} - \frac{35}{16} 10^{-6} + \ldots \right) \right]$$

so $\overline{P}_2$ can be represented adequately

$$\overline{P}_2 = \frac{3\mu_E \overline{R}_{EM}}{2R_{EM}^3} \alpha$$

Thus,

$$P_{2,j} = R_{EM,j} \frac{3\mu_E}{2R_{EM}^5} \left[2 \left(R_{EM,1} r_1 + R_{EM,2} r_2 + R_{EM,3} r_3 \right) + r_1^2 + r_2^2 + r_3^2 \right].$$

The effect of the higher order terms in the Earth's gravitational potential is obviously negligible.

The radiation pressure of the Sun at a distance of one astronomical unit is of the order of 4×10^{-5} dynes/m^2. With a LEM mass of 10 kg and a projected area of 10 m^2, the resultant acceleration is of the order of 10^{-12} m/sec^2, and may be neglected. Thus $\overline{P}_3 \doteq 0$.

The acceleration of the vehicle relative to the Moon due to the Sun's gravitational field is

$$\overline{P}_4 = \frac{\mu_s}{R_{SM}^2} \left[\frac{\overline{r}}{R_{SM}} + \frac{\overline{R}_{SM} + \overline{r}}{R_{SM}} \left(-\frac{3}{2}\alpha + \frac{15}{8}\alpha^2 - \dots \right) \right]$$

in which $\overline{R}_{MS}$ is the vector from the Sun to the Moon and $\alpha = r/R_{SM}$. Now μ_s/R_{SM}^2 is of the order of 10^{-11} m/sec^2, and the quantity in the brackets has magnitude less than unity, so for all practical purposes $\overline{P}_4 \doteq 0$.

The differential equation for the unperturbed motion of the vehicle is most easily solved in the manner indicated in Apollo Note No. 82.

The solution of the differential equation for the perturbation $\overline{\rho}$ must be obtained by numerical integration. The method of Runge and Kutta can be used for the first few points, and then the methods of Adams and Moulton. According to JPL TR 32-223, one minute intervals of time may be used for the integration in the vicinity of the Moon. The equation to be integrated is:

$$\ddot{\overline{\rho}} = \frac{-\mu_M}{R^3} \left[\overline{\rho} + \overline{R}\ F(Q) \right] + \overline{P}$$

in which the right-hand side is a function of time and position only.

This equation may be rewritten as a set of simultaneous first order differential equations by letting

$$\xi_1 \triangleq \rho_1$$

$$\xi_2 \triangleq \rho_2$$

$$\xi_3 \triangleq \rho_3$$

$$\xi_4 \triangleq \dot{\rho}_1$$

$$\xi_5 \triangleq \dot{\rho}_2$$

$$\xi_6 \triangleq \dot{\rho}_3$$

Then

$$\dot{\xi}_{i+3} = \frac{-\mu_M}{R^3}\left[\xi_i + R_i\, F(\Omega)\right] + P_i \qquad i = 1, 2, 3$$

$$\dot{\xi}_i = \xi_{i+3} \qquad i = 1, 2, 3$$

The initial conditions are

$$\xi_{i,\,0} = 0. \qquad i = 1, \ldots, 6$$

The solution is started using the Runge-Kutta method, using

$$\xi_{i,\,n+1} = \xi_{i,\,n} + (k_{i,\,1} + 2\,k_{i,\,2} + 2\,k_{i,\,3} + k_{i,\,4}) / 6$$

where:

$$k_{i,\,1} = h\,\dot{\xi}_i\,(T_n, \xi_{1,\,n}, \xi_{2,\,n}, \ldots, \xi_{6,\,n})$$

$$k_{i,\,2} = h\,\dot{\xi}_i\,(T_n + h/2,\ \xi_{1,\,n} + k_{1,\,1}/2, \ldots, \xi_{6,\,n} + k_{6,\,1}/2)$$

$$k_{i,\,3} = h\,\dot{\xi}_i\,(T_n + h/2,\ \xi_{1,\,n} + k_{1,\,2}/2, \ldots, \xi_{6,\,n} + k_{6,\,2}/2)$$

$$k_{i,\,4} = h\,\dot{\xi}_i\,(T_n + h, \xi_{1,\,n} + k_{1,\,3}, \ldots, \xi_{6,\,n} + k_{6,\,3})$$

in which

$$h \triangleq T_{n+1} - T_n$$

and is independent of n.

The steps of the Runge-Kutta method are repeated five times to obtain initial differences for use in the methods of Adams and Moulton. These differences are $\nabla^j \xi_{i,5}$ in which

$$\nabla w_n \triangleq w_n - w_{n-1}$$

$$\nabla^{m+1} w_n \triangleq \nabla^m w_n - \nabla^m w_{n-1} = \nabla \nabla^m w_n$$

In the next and succeeding iterations, the method of Adams is employed to obtain first estimates of $\xi_{i,n+1}$, and then the method of Moulton is applied repeatedly to obtain better values for these quantities. It is assumed that three iterations each time are sufficient.

In Adam's method

$$\xi_{i,n+1} = \xi_{i,n} + h\left[1 + \frac{1}{2}\nabla + \frac{5}{12}\nabla^2 + \frac{3}{8}\nabla^3 + \frac{251}{720}\nabla^4 + \frac{95}{288}\nabla^5\right]\xi_{i,n}$$

and in Moulton's method the improved value of $\xi_{i,n+1}$ is obtained from:

$$\text{Improved } \xi_{i,n+1} = \xi_{i,n} + h\left[1 - \frac{1}{2}\nabla - \frac{1}{12}\nabla^2 - \frac{1}{24}\nabla^3 - \frac{19}{720}\nabla^4 - \frac{3}{160}\nabla^5\right]\xi_{i,n+1}$$

After obtaining $\overline{\rho}$ and $\dot{\overline{\rho}}$ in this manner for all the points in time at which data is to be taken, the computed values of the observed quantities are determined. Considering only range rate from one station as an example, the expected number of Doppler cycles in the one minute of observation must be computed, allowing for the motions of the vehicle and station during the signal transit time.

Using a co-ordinate system fixed in the station, a signal received by the station at time T_n must have been transmitted by the vehicle τ_n seconds earlier, where

$$\tau_n = \left| \overline{s}\,(T_n - \tau_n) \right| / c$$

in which $\bar{s}\,(T_n - \tau_n)$ is the position of the vehicle relative to the station at time $T_n - \tau_n$ and c is the speed of light. If this signal is a re-transmittal of a signal the vehicle received from the station, it must have been transmitted by the station at time $T_n - 2\,\tau_n$. Any relativistic corrections necessitated by the vehicle off-setting the re-transmitted frequency are neglected here. τ_n is determined by solution of the equation:

$$c\,\tau_n = \left|\bar{s}\,(T_n - \tau_n)\right| = s\,(T_n - \tau_n)$$

In the vicinity of T_n, s (T) can be expressed as a quadratic in T. Let s_n denote s (T_n). Then,

$$s\,(T_n - \tau_n) = s_n - \frac{s_{n+1} - s_{n-1}}{2h}\,\tau_n + \frac{s_{n+1} - 2\,s_n + s_{n-1}}{2h^2}\,\tau_n^2$$

$$= s_n + b_n\,\tau_n + c_n\,\tau_n^2$$

and

$$c\,\tau_n = s_n + b_n\,\tau_n + c_n\,\tau_n^2$$

so

$$\tau_n = \frac{(c - b_n) - \sqrt{(c - b_n)^2 - 4\,c_n\,s_n}}{2c_n}$$

$$= \frac{c - b_n}{2\,c_n}\left[1 - \sqrt{1 - \frac{4\,c_n\,s_n}{(c - b_n)^2}}\right]$$

$$\cong \frac{c - b_n}{2\,c_n}\;\frac{2\,c_n\,s_n}{(c - b_n)^2}$$

$$\cong \frac{s_n}{c - b_n}$$

or, more accurately,

$$\tau_n = \frac{s_n}{c - \dot{s}_n}$$

Letting f be the transmitting frequency of the station, the number of cycles received by the station between T_n and T_{n+1} is

$f\left[(T_{n+1} - 2\tau_{n+1}) - (T_n - 2\tau_n)\right]$ and the number of Doppler cycles between this and the signal transmitted in the interval $T_{n+1} - T_n$ is $-2f(\tau_{n+1} - \tau_n)$.

Let

$$\Gamma_n \triangleq -2f(\tau_n - \tau_{n-1})$$

and let γ_n be the measured value of this quantity. Then

$$\sigma^2 = \frac{1}{N}\sum_{i=1}^{N}(\gamma_i - \Gamma_i)^2$$

is to be minimized by choice of the six orbit parameters $a_1, \ldots, a_6$. This is accomplished by computing Γ_i and $\partial\Gamma_i/\partial a_j$ for the assumed set of six orbit parameters, and setting $\partial\sigma^2/\partial a_j$ equal to zero. This gives six simultaneous linear equations for estimates of the changes in the parameters to minimize σ^2.

$$\sigma^2 = \frac{1}{N}\sum_{i=1}^{N}\left(\gamma_i - \Gamma_i - \frac{\partial\Gamma_i}{\partial a_1}\Delta a_1 - \ldots - \frac{\partial\Gamma_i}{\partial a_6}\Delta a_6\right)^2$$

$$\frac{\partial\sigma^2}{\partial\Delta a_j} = 0 = \frac{2}{N}\sum_{i=1}^{N}\left(\gamma_i - \Gamma_i - \frac{\partial\Gamma_i}{\partial a_1}\Delta a_1 - \ldots - \frac{\partial\Gamma_i}{\partial a_6}\Delta a_6\right)\frac{\partial\Gamma_i}{\partial a_j}$$

Letting

$$C_{jk} \triangleq \sum_{i=1}^{N} \frac{\partial \Gamma_i}{\partial a_j} \frac{\partial \Gamma_i}{\partial a_k}$$

and

$$e_k = \sum_{i=1}^{N} (\gamma_i - \Gamma_i) \frac{\partial \Gamma_i}{\partial a_k}$$

the above equations, $\partial \sigma^2 / \partial \Delta a_j = 0$, can be written as:

$$\left[C_{jk} \right] \left[\Delta a_j \right] = \left[e_k \right]$$

Inverting the C matrix, we find

$$\left[\Delta \alpha_j \right] = \left[C_{jk} \right]^{-1} \left[e_k \right].$$

The quantities $\partial \Gamma_i / \partial a_j$ to be used in the above equations are obtained as follows:

$$\Gamma_i = -2f (\tau_i - \tau_{i-1})$$

$$\frac{\partial \Gamma_i}{\partial a_j} = -2f \left(\frac{\partial \tau_i}{\partial a_j} - \frac{\partial \tau_{i-1}}{\partial a_j} \right)$$

Now,

$$\tau_i = \frac{s_i}{c - \dot{s}_i}$$

so

$$\frac{\partial \tau_i}{\partial a_j} = \frac{\tau_i}{s_i} \frac{\partial s_i}{\partial a_j} - \frac{\tau_i}{c - \dot{s}_i} \frac{\partial \dot{s}_i}{\partial a_j}$$

Thus $\partial \Gamma_i / \partial a_j$ can be computed. In calculating $\partial s_i / \partial a_j$ and $\partial \dot{s} / \partial a_j$, the unperturbed orbit may be used.

The orbit parameter estimates are changed in accordance with the Δa_j's obtained in this manner and the entire computation repeated.

The flow diagram for the computation could be as follows, although in practice a more efficient arrangement might be found. This flow diagram suffices for determining computation time.

1

Start

Insert h and "a priori" parameters $a_1, \ldots, a_6$

Note: Parameters are position and velocity relative to the Moon.

2 → Accept measurement γ_k

Is $k \geq 26$?

no → 2

yes

$\xi_{i,o} = 0 \qquad , \; i=1, \ldots, 6$

$R_o^2 = a_1^2 + a_2^2 + a_3^2$

$R_o = \sqrt{R_o^2}$

$V_o^2 = a_4^2 + a_5^2 + a_6^2$

$-2E = \frac{\mu_M}{R_o} - V_o^2$

$\sqrt{-2E} = \sqrt{-2E}$

Note: $\overline{H} = \overline{R_o} \times \overline{V_o}$

$H_1 = a_2 a_6 - a_3 a_5$

$H_2 = a_3 a_4 - a_1 a_6$

$$H_3 = a_1 a_5 - a_2 a_4$$

$$H^2 = H_1^2 + H_2^2 + H_3^2$$

$$H = \sqrt{H^2}$$

$$\hat{H}_1 = H_1/H$$

$$\hat{H}_2 = H_2/H$$

$$\hat{H}_3 = H_3/H$$

$$\hat{R}_1 = a_1/R_o$$

$$\hat{R}_2 = a_2/R_o$$

$$\hat{R}_3 = a_3/R_o$$

Note: $\hat{L} = \hat{R}_o \times \hat{H}$

$$\hat{L}_1 = \hat{R}_2 \hat{H}_3 - \hat{R}_3 \hat{H}_2$$

$$\hat{L}_2 = \hat{R}_3 \hat{H}_1 - \hat{R}_1 \hat{H}_3$$

$$\hat{L}_3 = \hat{R}_1 \hat{H}_2 - \hat{R}_2 \hat{H}_1$$

Note: In the $\hat{R}_o$, $\hat{H}$, $\hat{L}$ co-ordinate system the unperturbed orbit is in the $\hat{R}_o \hat{L}$ - plane.

$$H_\mu = H/\mu_M$$

$$H_{2\mu} = H^2/\mu_M$$

$$H_\nu = H_{2\mu}/R_o$$

$$e \cos \theta_o = H_\nu - 1$$

$$e \sin \theta_o = H_\mu (\hat{R}_1 a_4 + \hat{R}_2 a_5 + \hat{R}_3 a_6)$$

$$e^2 = (e \cos \theta_o)^2 + (e \sin \theta_o)^2$$

$$e = \sqrt{e^2}$$

$$(1 - e^2)^{1/2} = \sqrt{-2E} \quad H_\mu$$

$$e \sin \mathcal{E}_o = \frac{(1-e^2)^{1/2} (e \sin \theta_o)}{H_\nu}$$

$$e \cos \mathcal{E}_o = \frac{e^2 + (e \cos \theta_o)}{H_\nu}$$

$$\mathcal{E}_o = \tan^{-1} \frac{(e \sin \mathcal{E}_o)}{(e \cos \mathcal{E}_o)}$$

Note: Branch path necessary, but not shown, if $H_\nu = 1$. Also, quadrant decision necessary but not shown to determine quadrant of $\mathcal{E}_o$.

$$M_o = \mathcal{E}_o - (e \sin \mathcal{E}_o)$$

$$n = \frac{(-2E) \sqrt{-2E}}{\mu_M}$$

$$\alpha_5 = H/R_o$$

$$\alpha_4 = \sqrt{V_o^2 - \alpha_5^2}$$

$$\frac{\delta H}{\delta \alpha_j} = \delta_{1j} \alpha_5 - \delta_{2j} \alpha_4 + \delta_{5j} R_o \qquad j = 1, \ldots, 6$$

$$\frac{\delta E}{\delta \alpha_j} = \delta_{1j} \frac{\mu_M}{R_o^2} + \delta_{4j} \alpha_4 + \delta_{5j} \alpha_5$$

$$\frac{\partial e^2}{\partial \alpha_j} = 2 \frac{H^2}{\mu} \left[\frac{\partial E}{\partial \alpha_j} + \frac{2E}{H} \frac{\partial H}{\partial \alpha_j} \right]$$

$$\frac{\partial (e \cos \theta_o)}{\partial \alpha_j} = \frac{\alpha_5}{\mu} \left[2 \frac{\partial H}{\partial \alpha_j} - \delta_{1j} \alpha_5 \right]$$

$$\frac{\partial (e \sin \theta_o)}{\partial \alpha_j} = \frac{\alpha_5}{\mu} (\delta_{4j} R_o + \delta_{2j} \alpha_5) + \frac{\alpha_4}{\mu} \frac{\partial H}{\partial \alpha_j}$$

$$\frac{\partial (e \sin \mathcal{E}_o)}{\partial \alpha_j} = \frac{-(e \sin \mathcal{E}_o)}{2(1-e^2)} \frac{\partial e^2}{\partial \alpha_j} + \frac{(1-e^2)^{1/2}}{H_\nu} \frac{\partial (e \sin \theta_o)}{\partial \alpha_j} - \frac{(e \sin \mathcal{E}_o)}{H_\nu} \frac{\partial (e \cos \theta_o)}{\partial \alpha_j}$$

$$\frac{\partial (e \cos \mathcal{E}_o)}{\partial \alpha_j} = \frac{1}{H_\nu} \frac{\partial e^2}{\partial \alpha_j} + \left[\frac{(1-e)^{1/2}}{H_\nu} \right]^2 \frac{\partial (e \cos \theta_o)}{\partial \alpha_j}$$

$$\frac{\partial e}{\partial \alpha_j} = \cos \theta_o \frac{\partial (e \cos \theta_o)}{\partial \alpha_j} + \sin \theta_o \frac{\partial (e \sin \theta_o)}{\partial \alpha_j}$$

$$e \frac{\partial \mathcal{E}_o}{\partial \alpha_j} = \cos \mathcal{E}_o \frac{\partial (e \sin \mathcal{E}_o)}{\partial \alpha_j} - \sin \mathcal{E}_o \frac{\partial (e \cos \mathcal{E}_o)}{\partial \alpha_j}$$

$$e \frac{\partial M_o}{\partial \alpha_j} = (1 - e \cos \mathcal{E}_o) \left(e \frac{\partial \mathcal{E}_o}{\partial \alpha_j} \right) - \frac{\sin \mathcal{E}_o}{2} \frac{\partial e^2}{\partial \alpha_j}$$

$$\nu_o = \frac{\sin \mathcal{E}_o \left(\frac{1 - e \cos \mathcal{E}_o}{1 - e^2} + 1 \right) \frac{\partial e}{\partial \alpha_j}}{(1 - e \cos \mathcal{E}_o)^2}$$

i = 0

3

3

$$M_{i+1} = M_i + n\,h$$

$$\tilde{\mathcal{E}}_{i+1} = \mathcal{E}_i + \frac{n\,h}{1 - e \cos \mathcal{E}_i}$$

4 $\longrightarrow$ $$\tilde{M} = \tilde{\mathcal{E}}_{i+1} - e \sin \tilde{\mathcal{E}}_{i+1}$$

Is $\left| \dfrac{M_{i+1} - \tilde{M}}{1 - e \cos \tilde{\mathcal{E}}_i} \right| \leq K_1$?

yes

$$\mathcal{E}_{i+1} = \tilde{\mathcal{E}}_{i+1}$$

$$i = i + 1$$

$$\sin \mathcal{E}_i = \sin \mathcal{E}_i$$

$$\cos \mathcal{E}_i = \cos \mathcal{E}_i$$

$$\sin \theta_i = \frac{(1-e^2)^{1/2} \sin \mathcal{E}_i}{1 - e \cos \mathcal{E}_i}$$

$$\cos \theta_i = \frac{\cos \mathcal{E}_i - e}{1 - e \cos \mathcal{E}_i}$$

$$\sin(\theta_i - \theta_o) = \sin \theta_i \cos \theta_o - \sin \theta_o \cos \theta_i$$

$$\cos(\theta_i - \theta_o) = \cos \theta_i \cos \theta_o + \sin \theta_o \sin \theta_i$$

$$R_i = \frac{H_{2\mu}}{1 + e \cos \theta_i}$$

$$R_{i,R} = R_i \cos(\theta_i - \theta_o)$$

no

$$\tilde{\mathcal{E}}_{i+1} = \tilde{\mathcal{E}}_{i+1} + \left[\frac{M_{i+1} - \tilde{M}}{1 - e \cos \tilde{\mathcal{E}}_{i+1}} \right]$$

$\downarrow$ 4

$$R_{i,L} = R_i \sin(\theta_i - \theta_o)$$

$$R_{i,1} = R_{i,R}\,\hat{R}_1 + R_{i,L}\,\hat{L}_1$$

$$R_{i,2} = R_{i,R}\,\hat{R}_2 + R_{i,L}\,\hat{L}_2$$

$$R_{i,3} = R_{i,R}\,\hat{R}_3 + R_{i,L}\,\hat{L}_3$$

$$\dot{R}_i = \frac{e \sin \theta_i}{H_\mu}$$

$$R_i\,\dot{\theta}_i = \frac{H}{R_i^2}$$

$$\dot{R}_{i,R} = \dot{R}_i \cos(\theta_i - \theta_o) - R_i\,\dot{\theta}_i \sin(\theta_i - \theta_o)$$

$$\dot{R}_{i,L} = \dot{R}_i \sin(\theta_i - \theta_o) + R_i\,\dot{\theta}_i \cos(\theta_i - \theta_o)$$

$$\dot{R}_{i,1} = \dot{R}_{i,R}\,\hat{R}_1 + \dot{R}_{i,L}\,\hat{L}_1$$

$$\dot{R}_{i,2} = \dot{R}_{i,R}\,\hat{R}_2 + \dot{R}_{i,L}\,\hat{L}_2$$

$$\dot{R}_{i,3} = \dot{R}_{i,R}\,\hat{R}_3 + \dot{R}_{i,L}\,\hat{L}_3$$

$i \geq k$

no → 3

yes

$\xi_{j,o} = 0 \qquad j = 1, \ldots, 6$

$i = 0$

$\ell = 1$

5

5

$$r_{1,i} = R_{1,i} + \xi_{1,i}$$

$$r_{2,i} = R_{2,i} + \xi_{2,i}$$

$$r_{3,i} = R_{3,i} + \xi_{3,i}$$

$$r_i^2 = r_{1,i}^2 + r_{2,i}^2 + r_{3,i}^2$$

$$r_i = \sqrt{r_i^2}$$

$$I = \frac{\left(A_1 r_{1,i}^2 + A_2 r_{2,i}^2 + A_3 r_{3,i}^2\right)}{r_i^2}$$

$$\frac{3G}{2r_i^5} = \frac{3G}{2} \cdot \frac{1}{(r_i^2)^2 r_i}$$

$$A = A_1 + A_2 + A_3 - 5\,I$$

$$P_{1,1} = (A - A_1)\ \frac{3G}{2r_i^5}\ \xi_1$$

$$P_{1,2} = (A - A_2)\ \frac{3G}{2r_i^5}\ \xi_2$$

$$P_{1,3} = (A - A_3)\ \frac{3G}{2r_i^5}\ \xi_3$$

Look up $\overline{R}_{EM}$ in Ephemeris (assume 200 multiplications and 200 additions for each look-up and co-ordinate rotations).

$$R^2_{EM} = R^2_{EM,1} + R^2_{EM,2} + R^2_{EM,3}$$

$$\overline{R}_{EM} \cdot \overline{r} = R_{EM,1}\, r_{1,i} + R_{EM,2}\, r_{2,i} + R_{EM,3}\, r_{3,i}$$

$$p_2 = \frac{3\mu_E}{2\left(R^2_{EM}\right)^2 \sqrt{R^2_{EM}}} \left(\overline{R}_{EM} \cdot \overline{r} + r_i^2\right)$$

$$P_1 = P_{1,1} + p_2\, R_{EM,1}$$

$$P_2 = P_{1,2} + p_2\, R_{EM,2}$$

$$P_3 = P_{1,3} + p_2\, R_{EM,3}$$

$$Q = \frac{\xi_1\,(R_1 + \xi_1/2) + \xi_2\,(R_2 + \xi_2/2) + \xi_3\,(R_3 + \xi_3/2)}{R^2}$$

$$F(Q) = 3\,Q\,(1 - \frac{5}{2}\,Q)$$

$$\dot{\xi}_1 = \xi_4$$

$$\dot{\xi}_2 = \xi_5$$

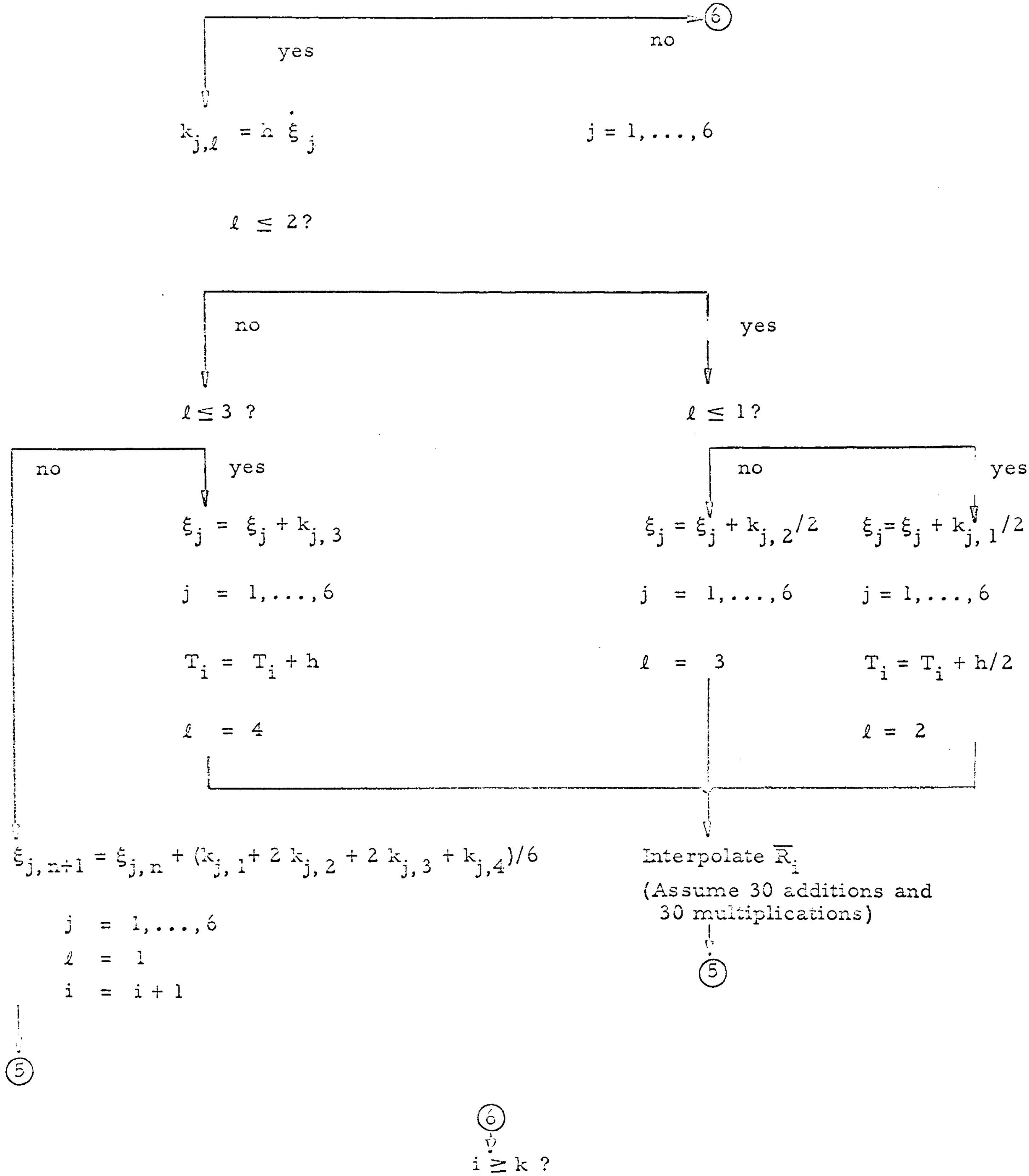
$$\dot{\xi}_3 = \xi_6$$

$$\dot{\xi}_4 = \frac{-\mu_M}{R^3}\left[\xi_1 + R_1\, F(Q)\right] + P_1$$

$$\dot{\xi}_5 = \frac{-\mu_M}{R^3}\left[\xi_2 + R_2\, F(Q)\right] + P_2$$

$$\dot{\xi}_6 = \frac{-\mu_M}{R^3}\left[\xi_3 + R_3\, F(Q)\right] + P_3$$

$i \leq 5$?


yes
no
⑥
$k_{j,\ell} = h\,\dot{\xi}_j$
$j = 1, \ldots, 6$
$\ell \leq 2$?
no
yes
$\ell \leq 3$?
$\ell \leq 1$?
no
yes
no
yes
$\xi_j = \xi_j + k_{j,3}$
$j = 1, \ldots, 6$
$T_i = T_i + h$
$\ell = 4$
$\xi_j = \xi_j + k_{j,2}/2$
$j = 1, \ldots, 6$
$\ell = 3$
$\xi_j = \xi_j + k_{j,1}/2$
$j = 1, \ldots, 6$
$T_i = T_i + h/2$
$\ell = 2$
$\xi_{j,n+1} = \xi_{j,n} + (k_{j,1} + 2\,k_{j,2} + 2\,k_{j,3} + k_{j,4})/6$
$j = 1, \ldots, 6$
$\ell = 1$
$i = i + 1$
⑤
Interpolate $\overline{R}_i$
(Assume 30 additions and 30 multiplications)
⑤
⑥
$i \geq k$?

no $\longrightarrow$ yes ⑦

$$\xi_{j,i+1} = \xi_{j,i} + h\left[\gamma_o\,\xi_{j,i} + \gamma_1\xi_{j,i-1} + \gamma_2\,\xi_{j,i-2} + \gamma_3\,\xi_{j,i-3}\right.$$

$$\left. + \gamma_4\,\xi_{j,i-4} + \gamma_5\,\xi_{j,i-5}\right] \qquad j=1,\ldots,6$$

$m = 0$

⑧ $\rightarrow$ $$\xi_{j,i+1} = \xi_{j,i+1} + h\left[\beta_o\,\xi_{j,i+1} + \beta_1\xi_{j,i} + \beta_2\,\xi_{j,i-1} + \beta_3\,\xi_{j,i-2}\right.$$

$$\left. + \beta_4\,\xi_{j,i-3} + \beta_5\,\xi_{j,i-4}\right] \qquad j=1,\ldots,6$$

$m = m + 1$

$m \geq 3$?

no $\rightarrow$ ⑧

yes

⑤

⑦

$i = 0$

⑨ $\rightarrow$ $R_{ED,I} = \zeta_1 \cos(\omega_e T_i + \ell_o)$

$R_{ED,II} = \zeta_1 \sin(\omega_e T_i + \ell_o)$

$R_{ED,III} = \zeta_2$

$$R_{ED,1} = C_{11} R_{ED,I} + C_{12} R_{ED,II} + C_{13} R_{ED,III}$$

$$R_{ED,2} = C_{21} R_{ED,I} + C_{22} R_{ED,II} + C_{23} R_{ED,III}$$

$$R_{ED,3} = C_{31} R_{ED,I} + C_{32} R_{ED,II} + C_{33} R_{ED,III}$$

$$\dot{R}_{ED,I} = - \omega_e R_{ED,II}$$

$$\dot{R}_{ED,II} = \omega_e R_{ED,I}$$

$$\dot{R}_{ED,III} = 0$$

$$\dot{R}_{ED,1} = C_{11} \dot{R}_{ED,I} + C_{12} \dot{R}_{ED,II} + C_{13} \dot{R}_{ED,III}$$

$$\dot{R}_{ED,2} = C_{21} \dot{R}_{ED,I} + C_{22} \dot{R}_{ED,II} + C_{23} \dot{R}_{ED,III}$$

$$\dot{R}_{ED,3} = C_{31} \dot{R}_{ED,I} + C_{32} \dot{R}_{ED,II} + C_{33} \dot{R}_{ED,III}$$

$\dot{R}_{EM,1}$, $\dot{R}_{EM,2}$, $\dot{R}_{EM,3}$ from computed values of $R_{EM,1}$, $R_{EM,2}$, $R_{EM,3}$ by finite differences. (Assume 10 multiplications and 10 additions for each $\dot{R}_{EM,j}$).

$$\dot{r}_1 = \dot{R}_{i,1} + \dot{\xi}_4$$

$$\dot{r}_2 = \dot{R}_{i,2} + \dot{\xi}_5$$

$$\dot{r}_3 = \dot{R}_{i,3} + \dot{\xi}_6$$

$$s_{1,i} = R_{EM,1} + r_1 - R_{ED,1}$$

$$s_{2,i} = R_{EM,2} + r_2 - R_{ED,2}$$

$$s_{3,i} = R_{EM,3} + r_3 - R_{ED,3}$$

$$s^2 = s_{1,i}^2 + s_{2,i}^2 + s_{3,i}^2$$

$$s_i = \sqrt{s^2}$$

$$\dot{s}_{1,i} = \dot{R}_{EM,1} + \dot{r}_1 - \dot{R}_{ED,1}$$

$$\dot{s}_{i,2} = \dot{R}_{EM,2} + \dot{r}_2 - \dot{R}_{ED,2}$$

$$\dot{s}_{i,3} = \dot{R}_{EM,3} + \dot{r}_3 - \dot{R}_{ED,3}$$

$$s\,\dot{s} = s_{1,i}\,\dot{s}_{1,i} + s_{2,i}\,\dot{s}_{2,i} + s_{3,i}\,\dot{s}_{3,i}$$

$$\dot{s}_i = \frac{1}{s_i}(s\,\dot{s})$$

$$\tau_i = \frac{s_i}{c - s_i}$$

$i = i + 1$

$i > k$?

yes → ↓ no → 9

$i = 0$

10

10

$$\gamma_i - \Gamma_i = \gamma_i - 2f(\tau_i - \tau_{i-1})$$

$$\sum(\gamma_i - \Gamma_i)^2 = \sum(\gamma_i - \Gamma_i)^2 + (\gamma_i - \Gamma_i)^2$$

$$e\frac{\partial\theta_i}{\partial\alpha_j} = \frac{(1-e^2)^{1/2}}{(1-e\cos\mathcal{E}_i)^2}\left\{\frac{3(M_i-M_o)}{2E}\quad e\frac{\partial E}{\partial\alpha_j} + \left(e\frac{\partial M_o}{\partial\alpha_j}\right) + \frac{\sin\mathcal{E}_i}{2}\left(\frac{1-e\cos\mathcal{E}_i}{1-e^2}+1\right)\frac{\partial e^2}{\partial\alpha_j}\right\}$$

$$\frac{\partial(\theta_i-\theta_o)}{\partial\alpha_j} = (1-e^2)^{1/2}\left\{\begin{array}{l}\dfrac{3(M_i-M_o)}{2E}\dfrac{\partial E}{\partial\alpha_j} + \sin\mathcal{E}_i\left(\dfrac{1-e\cos\mathcal{E}_i}{1-e^2}+1\right)\dfrac{\partial e}{\partial\alpha_j}\\ -\nu_o\\ +\dfrac{(\cos\mathcal{E}_i-\cos\mathcal{E}_o)\left(2-e\left[\cos\mathcal{E}_i+\cos\mathcal{E}_o\right]\right)}{(1-e\cos\mathcal{E}_i)^2\,(1-e\cos\mathcal{E}_o)^2}\left(e\dfrac{\partial M_o}{\partial\alpha_j}\right)\end{array}\right\}$$

$$\frac{\partial R_i}{\partial\alpha_j} = \frac{2R_i}{H}\frac{\partial H}{\partial\alpha_j} + \frac{R_i}{1+e\cos\theta_i}\left[\cos\theta_i\frac{\partial e}{\partial\alpha_j} - \sin\theta_i\left(e\frac{\partial\theta_i}{\partial\alpha_j}\right)\right]$$

$$\frac{\partial R_{i,R}}{\partial\alpha_j} = -R_{i,L}\frac{\partial(\theta_i-\theta_o)}{\partial\alpha_j} + \cos(\theta_i-\theta_o)\frac{\partial R_i}{\partial\alpha_j}$$

$$\frac{\partial R_{i,L}}{\partial \alpha_j} = R_{i,R} \frac{\partial(\theta_i - \theta_o)}{\partial \alpha_j} + \sin(\theta_i - \theta_o) \frac{\partial R_i}{\partial \alpha_j}$$

$$\frac{\partial R_{i,H}}{\partial \alpha_j} = \delta_{3j} \frac{\alpha_5 R_{i,R} - \alpha_4 R_{i,L}}{H} + \delta_{6j} \frac{R_{iL}}{\alpha_5}$$

$$\frac{\partial R_{i,1}}{\partial \alpha_j} = \frac{\partial R_{i,R}}{\partial \alpha_j} \hat{R}_1 + \frac{\partial R_{i,L}}{\partial \alpha_j} \hat{L}_1 + \frac{\partial R_{i,H}}{\partial \alpha_j} \hat{H}_1$$

$$\frac{\partial R_{i,2}}{\partial \alpha_j} = \frac{\partial R_{i,R}}{\partial \alpha_j} \hat{R}_2 + \frac{\partial R_{i,L}}{\partial \alpha_j} \hat{L}_2 + \frac{\partial R_{i,H}}{\partial \alpha_j} \hat{H}_2$$

$$\frac{\partial R_{i,3}}{\partial \alpha_j} = \frac{\partial R_{i,R}}{\partial \alpha_j} \hat{R}_3 + \frac{\partial R_{i,L}}{\partial \alpha_j} \hat{L}_3 + \frac{\partial R_{i,H}}{\partial \alpha_j} \hat{H}_3$$

$$A = \frac{1}{R_i} \frac{\partial H}{\partial \alpha_j} - \frac{H}{R_i^2} \frac{\partial R_i}{\partial \alpha_j} + \dot{R}_i \frac{\partial(\theta_i - \theta_o)}{\partial \alpha_j}$$

$$\frac{\partial \dot{R}_i}{\partial \alpha_j} = \frac{1}{H_\mu} \left[\sin\theta_i \frac{\partial e}{\partial \alpha_j} + \cos\theta_i \left(e \frac{\partial \theta_i}{\partial \alpha_j} \right) \right] - \frac{\dot{R}_i}{H} \frac{\partial H}{\partial \alpha_j}$$

$$B = -\frac{H}{R_i} \frac{\partial(\theta_i - \theta_o)}{\partial \alpha_j} + \frac{\partial \dot{R}_i}{\partial \alpha_j}$$

$$\frac{\partial \dot{R}_{i,R}}{\partial \alpha_j} = -A \sin(\theta_i - \theta_o) + B \cos(\theta_i - \theta_o)$$

$$\frac{\partial \dot{R}_{i,L}}{\partial \alpha_j} = A \cos(\theta_i - \theta_o) + B \sin(\theta_i - \theta_o)$$

$$\frac{\partial \dot{R}_{i,H}}{\partial \alpha_j} = \delta_{3j} \quad \frac{\alpha_5 \dot{R}_{i,R} - \alpha_4 \dot{R}_{i,L}}{H} + \delta_{6j} \quad \frac{\dot{R}_{i,L}}{\alpha_5}$$

$$\frac{\partial \dot{R}_{i,1}}{\partial \alpha_j} = \frac{\partial \dot{R}_{i,R}}{\partial \alpha_j} \hat{R}_1 + \frac{\partial \dot{R}_{i,L}}{\partial \alpha_j} \hat{L}_1 + \frac{\partial \dot{R}_{i,H}}{\partial \alpha_j} \hat{H}_1$$

$$\frac{\partial \dot{R}_{i,2}}{\partial \alpha_j} = \frac{\partial \dot{R}_{i,R}}{\partial \alpha_j} \hat{R}_2 + \frac{\partial \dot{R}_{i,L}}{\partial \alpha_j} \hat{L}_2 + \frac{\partial \dot{R}_{i,H}}{\partial \alpha_j} \hat{H}_2$$

$$\frac{\partial \dot{R}_{i,3}}{\partial \alpha_j} = \frac{\partial \dot{R}_{i,R}}{\partial \alpha_j} \hat{R}_3 + \frac{\partial \dot{R}_{i,L}}{\partial \alpha_j} \hat{L}_3 + \frac{\partial \dot{R}_{i,H}}{\partial \alpha_j} \hat{H}_3$$

$$\frac{\partial s}{\partial \alpha_j} = \frac{s_{1,i} \dfrac{\partial R_{i,1}}{\partial \alpha_j} + s_{2,i} \dfrac{\partial R_{i,2}}{\partial \alpha_j} + s_{3,i} \dfrac{\partial R_{i,3}}{\partial \alpha_j}}{s}$$

$$\frac{\partial \dot{s}}{\partial \alpha_j} = \frac{1}{s}\left[s_{1,i} \frac{\partial \dot{R}_{i,1}}{\partial \alpha_j} + s_{2,i} \frac{\partial \dot{R}_{i,2}}{\partial \alpha_j} + s_{3,i} \frac{\partial \dot{R}_{i,3}}{\partial \alpha_j} - \dot{s} \frac{\partial s}{\partial \alpha_j} \right.$$

$$\left. + \dot{s}_{i,1} \frac{\partial R_{i,1}}{\partial \alpha_j} + \dot{s}_{i,2} \frac{\partial R_{i,2}}{\partial \alpha_j} + \dot{s}_{i,3} \frac{\partial R_{i,3}}{\partial \alpha_j} \right]$$

$$\frac{\partial \tau_i}{\partial \alpha_j} = \frac{\tau_i}{s_i} \frac{\partial s_i}{\partial \alpha_j} - \frac{\tau_i}{c-\dot{s}_i} \frac{\partial \dot{s}_i}{\partial \alpha_j}$$

$i=0$?

no

$$\phi_{i,j} = \frac{\partial \tau_i}{\partial \alpha_j} - \frac{\partial \tau_{i-1}}{\partial \alpha_j}$$

$$\frac{1}{4f^2} \sum \frac{\partial \Gamma_i}{\partial \alpha_j} \frac{\partial \Gamma_i}{\partial \alpha_k} = \frac{1}{4f^2} \sum \frac{\partial \Gamma_i}{\partial \alpha_j} \frac{\partial \Gamma_i}{\partial \alpha_k} + \phi_{i,j}\, \phi_{i,k}$$

$$-\frac{1}{2f^2} \sum (\gamma_i - \Gamma_i) \frac{\partial \Gamma_i}{\partial \alpha_j} = -\frac{1}{2f^2} \sum (\gamma_i - \Gamma_i) \frac{\partial \Gamma_i}{\partial \alpha_j} + (\gamma_i - \Gamma_i) \frac{\partial \Gamma_i}{\partial \alpha_j}$$

$i = i+1$

$i > k$?

yes — 11

no — 10

yes

$$\sum (\gamma_i - \Gamma_i)^2 = 0$$

$$\frac{1}{4f^2} \sum \frac{\partial \Gamma_i}{\partial \alpha_j} \frac{\partial \Gamma_i}{\partial \alpha_k} = 0$$

$$-\frac{1}{2f} \sum (\gamma_i - \Gamma_i) \frac{\partial \Gamma_i}{\partial \alpha_j} = 0$$

10

$$C_{jk} = 4f^2 \left(\frac{1}{4f^2} \sum \frac{\partial \Gamma_i}{\partial \alpha_j} \frac{\partial \Gamma_i}{\partial \alpha_k} \right)$$

$$e_k = -2f \left[\frac{-1}{2f} \sum (\gamma_i - \Gamma_i) \frac{\partial \Gamma_i}{\partial \alpha_k} \right]$$

Compute $\left[F_{kj} \right] = \left[C_{jk} \right]^{-1}$ using Jordan-Gauss method (assume 532 multiplications, 1064 additions, 72 divisions)

$$\Delta \alpha_j = \sum_{k=1}^{6} F_{kj} e_k$$

End

Computations Required

	Additions Subtraction	Multiplications	Divisions	Roots, Trigonometric	Load, Store
1 → 2	0	0	0	0	0
2 → 3	90	163	32	6	150
3 → 4	60	0	30	0	60
4 → 5	24	33	5	5	20
5 → 6	8,560	7,500	120	60	8,000
6 → 7	3,060	2,520	0	0	3,000
7 → 10	2,430	220	60	180	1,000
10 → 11	11,460	12,960	870	0	11,000
11 → End	1,100	610	72		1,000
	26,784	34,006	1,189	251	24,230

The following mean times are allowed for the indicated double precision operations in the IBM 7094 computer (μ sec)

15	15	20	350	5

The total computation time is then 1.15 seconds. Allowing a factor of 2.5 for bookkeeping --indirect addressing, etc. -- the total computation time is still less than 3 seconds. Thus it appears that course correction instructions certainly can be given less than a minute after the last observation.

If range, range-rate and angle from three stations are to be used instead of just range rate from one, it is estimated that the following changes in computations will be needed.

	Additions Subtractions	Multiplications	Divisions	Roots, Trigonometric	Load, Store
1 → 7	0	0	0	0	0
7 → 10	7,300	700	200	600	3,000
10 → 11	20,000	24,000	1,500	0	20,000
11 → End	13,400	6,700	550	0	10,000
	40,700	31,400	2,250	600	33,000

The additional computing time base on mean IBM 7094 times is 1.51 seconds. Allowing a factor of 2.5 this becomes 3.8 seconds.

Thus, even for range, range-rate and angle from three stations the total computation time is of the order of 7 seconds --far less than a minute.

APOLLO NOTE NO. 95

H. Engel
15 August 1963

GROUND ASSISTANCE TO LEM, INCLUDING MID-COURSE CORRECTION

Assume that at time $T = 0$ a priori orbit data are available and observations start. Assume further that at time T_1 the vehicle boosts for a mid-course correction based on the data up to that time, and the variance of the boost is known. The vehicle is observed until T_2, at which time the covariance matrix of the errors in the orbit parameters is calculated.

Let C_j denote the j-th information matrix and V_j the j-th covariance matrix, $C_j V_j = I$.

Let V_o be the covariance matrix of errors in initial position and velocity of the a priori data, and C_1 the corresponding information matrix of observations up to time T_1. Then

$$C_o = V_o^{-1}$$

$$C_2 = C_o + C_1$$

and

$$V_2 = C_2^{-1}$$

in which V_2 is the covariance matrix of errors in initial position and velocity resulting from the use of both sets of data.

The corresponding covariance matrix V_3 of errors in position and velocity at time T_1 is

$$V_3 = \mathcal{Q}_1 V_2 \mathcal{Q}_1^T$$

in which $\mathcal{Q}_1$ is the Jacobian of the components at time T_1 with respect to those at time 0.

Assume that at time T_1 a boost with zero mean and covariance matrix V_4 is applied. In the simplest case, all the components of this matrix are zero except for the three main diagonal element corresponding to the velocity components, which are $\sigma_{\dot{r}}^2$.

The covariance matrix of errors in position and velocity after boost at time T_1 is

$$V_5 = V_3 + V_4$$

The corresponding covariance matrix at time zero is

$$V_6 = Q_1^{-1} \; V_5 \, Q_1^{-1T}$$

and

$$C_6 = V_6^{-1}$$

The information matrix of the observations from T_1 to T_2 is C_7. The information matrix from combining this with previous observations is C_8.

$$C_8 = C_6 + C_7$$

and the corresponding covariance matrix is

$$V_8 = C_8^{-1}$$

The covariance matrix of errors in position and velocity at time T_2 is V_9

$$V_9 = Q_2 \, V_8 \, Q_2^T$$

in which Q_2 is the Jacobian of the components at time T_2 with respect to those at time zero.

Combining all these intermediate stages, we find

$$C_o = V_o^{-1}$$

$$V_3 = Q_1 \left[C_o + C_1\right]^{-1} Q_1^T$$

$$C_6 = Q_1^T \left[V_3 + V_4 \right]^{-1} Q_1$$

$$V_9 = Q_2 \left[C_6 + C_7 \right]^{-1} Q_2^T$$

APOLLO NOTE NO. 96

L. Horowitz
15 August 1963

USE OF LEM/CM OBSERVATIONS

L. Lustick has pointed out a simple way to determine the information matrix (inverse of covariance matrix) resulting from observations of the LEM by the CM: If the CM orbit is circular, the necessary computations are the same as for an observing station on the surface of the earth (Apollo Note No. 82) with the earth-moon distance reduced to zero, the station latitude zero, the radius of the earth changed to the radius of the CM orbit, and the angular rate of the earth set equal to the angular rate of the CM in its orbit. The calculations may be performed for the same kinds of observables as already done for observations from earth—range, range rate and angle.

Some care is necessary in selecting angles to be used. For the nominal ascent (Hohmann transfer), the orbits of the CM and LEM are coplanar. Then a mechanization for defining the relative orientations of the three coordinate systems X, X' and $\widetilde{X}$ of Apollo Note No. 82 is simply to take all three systems coincident. Thus, X' is a right-handed, non-rotating, moon-centered coordinate system in which x' is along the initial position vector $(x'_o, 0, 0)$ of the LEM, vehicle motion is in the x'-y' plane, and y' is directed so that $\dot{y}'_o$ is positive. $\widetilde{X}$ and X are then taken to coincide with X' so that $(\widetilde{x}, \widetilde{y}, \widetilde{z}) = (x, y, z) = (x', y', z')$. We introduce the initial central angle α between the two vehicles as what was originally the DSIF station longitude in the X system. This mechanization treats the CM as a "pseudo-DSIF" to permit full utilization of the existing computer program for calculating information matrices. In this formulation, the rotation matrices L and K transforming between the 3 coordinate systems will reduce to the identity matrix. The resulting input data for the program can then be taken as follows:

$\beta = 0$ (since no earth involved)

$XI = 0$

$\eta = 0$

$ZETA = 0$

$\lambda = 0$ (latitude of DSIF)

α = central angle between LEM and CM, where α is positive for CM leading LEM

$$\omega_e = \omega_{cm} = \sqrt{\frac{\mu}{R^3}} = 8.5463 \times 10^{-4} \text{ rad/sec}$$

based upon: altitude = 80 n. mi.,

$R_m = 1.7373 \times 10^3$ km and $\mu = 4.896 \times 10^3 \ \mathrm{km^3/sec^2}$

$\omega_m = 0$ (not relevant since $\rho_m \longrightarrow 0$)

ρ_e = moon radius + 80 n. mi. = 1.8855×10^6 meters

$\rho_m = 0$ (since earth-moon distance = 0)

$\mu = 4.896 \times 10^{12} \ \mathrm{m^3/sec^2}$

$$\left.\begin{array}{l} L = 0 \\ \ell = 0 \end{array}\right\} \text{latitude and longitude of sublunar point}$$

The initial values for nominal lunar trajectories can then be selected as desired.

APOLLO NOTE NO. 99

H. Engel
13 September 1963

PRELIMINARY RESULTS OF COMPUTER ANALYSES

Introduction

A problem susceptible to verification by a separate analysis has been run, and indicates that the computer program is correct.

In addition, a number of successful computer runs to determine the errors in future vehicle position have been accomplished.

These are all reported in this note.

Check Problem

The problem of determining vehicle position and velocity and probable errors in these quantities on the basis of one range and one range-rate measurement from each of three stations simultaneously is capable of direct analytic solution, and serves as a check on the computer program and on the more elaborate analysis on which it depends.

The distance s_i from station i to the vehicle is given by:

$$s_i^2 = \sum_{j=1}^{3} (x_j - x_{ji})^2$$

in which (x_1, x_2, x_3) is the vehicle position and (x_{1i}, x_{2i}, x_{3i}) is the position of the i-th station. Following Apollo Note No. 87,

$$\frac{\delta s_i}{\delta x_j} = \frac{1}{s_i} (x_j - x_{ji})$$

$$\frac{\delta \dot{s}_i}{\delta x_j} = \frac{\dot{x}_j}{s} - \frac{\dot{s}_i}{s_i} \frac{\delta s_i}{\delta x_j}$$

and

$$\frac{\partial \dot{s}_i}{\partial \dot{x}_j} = \frac{\partial s_i}{\partial x_j}$$

Now consider a configuration of vehicle and stations such as shown in Figure 1, which corresponds to a possible situation with the vehicle orbiting the Moon. The stations are all in the yz-plane, at the vertices of an equilateral triangle centered on the origin. The vehicle is on the x axis at a distance ℓ from the origin. ℓ is equal to the Earth-Moon distance less the Earth radius, less the Moon radius. The vehicle has a velocity

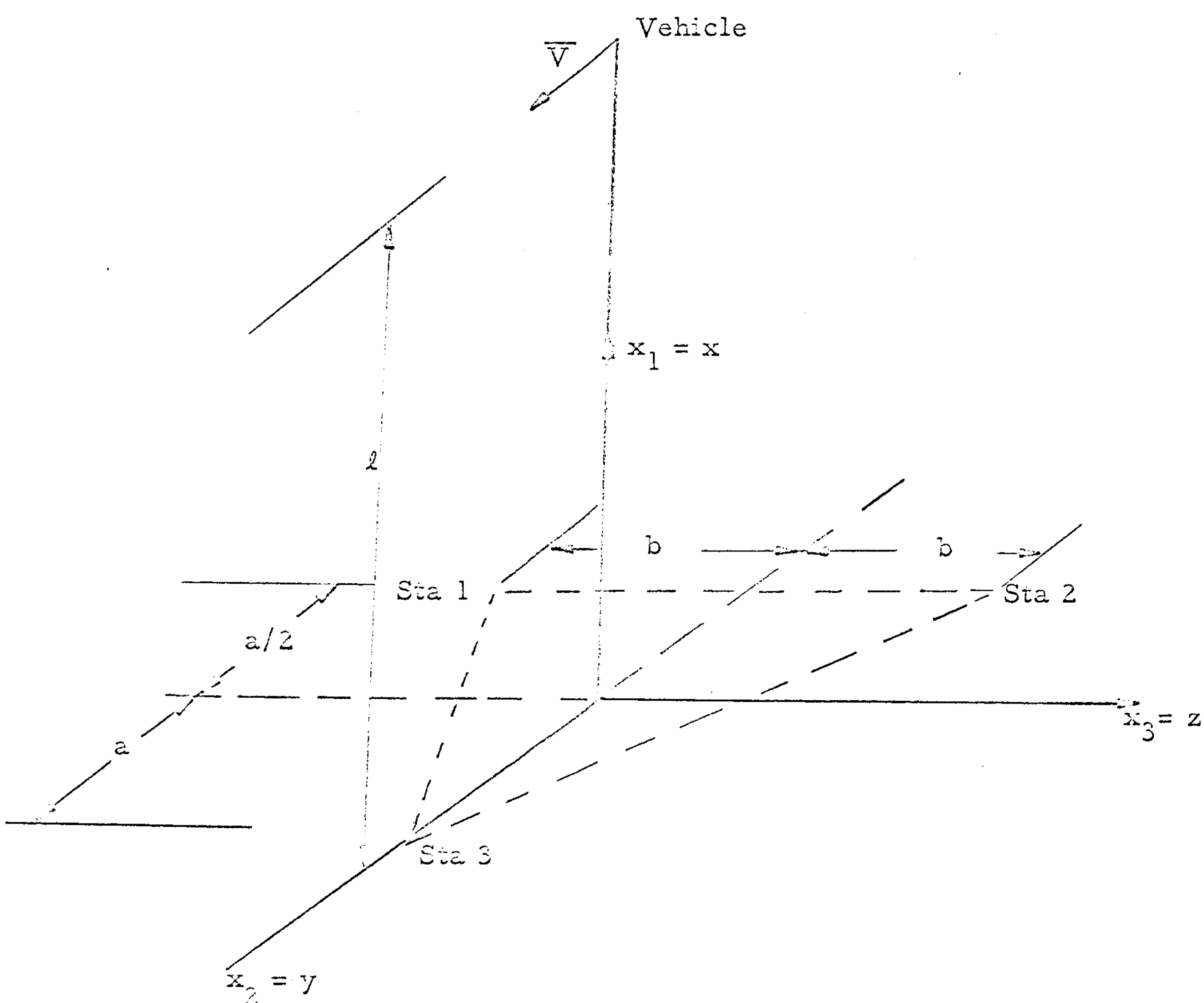


Figure 1.

V equal to 1.7017×10^3 m/sec in the y direction.

$$s = s_1 = s_2 = s_3$$

$$\overline{V} = 0\,\hat{x} + V\,\hat{y} + 0\,\hat{z}$$

$$\hat{s}_1 = \frac{\ell}{s}\,\hat{x} - \frac{a}{s}\,\hat{y} + 0\,\hat{z}$$

$$\hat{s}_2 = \frac{\ell}{s}\,\hat{x} + \frac{a}{2s}\,\hat{y} - \frac{b}{s}\,\hat{z}$$

$$\hat{s}_3 = \frac{\ell}{s}\,\hat{x} + \frac{a}{2s}\,\hat{y} + \frac{b}{s}\,\hat{z}$$

Then,

$$\dot{s}_i = \overline{V} \cdot \hat{s}_i$$

$$\dot{s}_1 = -\frac{aV}{s}$$

$$\dot{s}_2 = \frac{aV}{2s}$$

$$\dot{s}_3 = \frac{aV}{2s}$$

a_1 through a_6 are the parameters that describe the vehicle orbit in the computer calculations. a_1 through a_3 are vehicle position relative to the Moon, and a_4 through a_6 are vehicle velocity relative to the Moon the direction of a_1 and a_4 is $-\hat{x}$

direction of a_2 and a_5 is $\hat{y}$

direction of a_3 and a_6 is $-\hat{z}$.

Then the partial derivatives of s_i and $\dot{s}_i$ with respect to the vehicle orbit parameters are:

$$
\frac{\partial s_i}{\partial a_j} = \begin{bmatrix} -\ell/s & -\ell/s & -\ell/s \\ a/(2s) & a/(2s) & -a/s \\ -b/s & b/s & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
$$

$$
\frac{\partial \dot{s}_i}{\partial a_j} = \begin{bmatrix} aV\ell/(2s^3) & aV\ell/(2s^3) & -aV\ell/s^3 \\ \frac{V}{s}\left(1-\left[\frac{a}{2s}\right]^2\right) & \frac{V}{s}\left(1-\left[\frac{a}{2s}\right]^2\right) & \frac{V}{s}\left(1-\left[\frac{a}{s}\right]^2\right) \\ abV/(2s^3) & -abV/(2s^3) & 0 \\ -\ell/s & -\ell/s & -\ell/s \\ a/(2s) & a/(2s) & -a/s \\ -b/s & b/s & 0 \end{bmatrix}
$$

The information matrix corresponding to range measurements from station i has components

$$
C_{jk}^{i,R} = \frac{\partial s_i}{\partial a_j} \frac{\partial s_i}{\partial a_k}
$$

The corresponding information matrix elements for range-rate measurements are:

$$C^{i,\dot{R}}_{jk} = \frac{\partial \dot{s}_i}{\partial a_j} \frac{\partial \dot{s}_i}{\partial a_j}$$

All may be calculated from the $\partial s_i/\partial a_j$ and $\partial \dot{s}_i/\partial a_j$ matrices already given. We show, below, these $C^{i,R}$ and $C^{i,\dot{R}}$ matrices as computed in this fashion together with the values determined from the computer program (in parentheses). The small differences can be attributed to slightly different geometrics employed in the two calculations.

$$C^{1,R}_{jk} = \begin{bmatrix}
(1.0+0) & (-.41-2) & (.78-2) & (-.20-14) & (-.91-12) & (0) \\
1.0+0 & -.41-2 & .78-2 & 0 & 0 & 0 \\
 & & & & & \\
(.41-2) & (.17-4) & (-.32-4) & (.82-17) & (.37-14) & (0) \\
-.41-2 & .16-4 & -.32-4 & 0 & 0 & 0 \\
 & & & & & \\
(.78-2) & (-.32-4) & (.61-4) & (-.16-16) & (-.71-14) & (0) \\
0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & \\
(-.20-14) & (.82-17) & (-.16-16) & (.40-29) & (.18-26) & (0) \\
0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & \\
(-.91-12) & (.37-14) & (-.71-14) & (.18-26 & (.82-24) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & \\
(0) & (0) & (0) & (0) & (0) & (0) \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

$$C_{jk}^{2,R} = \begin{bmatrix}
\begin{matrix}(1.0+0)\\ 1.0+0\end{matrix} & \begin{matrix}(-.41-2)\\ -.45-2\end{matrix} & \begin{matrix}(-.78-2)\\ -.78-2\end{matrix} & \begin{matrix}(-.20-14)\\ 0\end{matrix} & \begin{matrix}(-.91-12)\\ 0\end{matrix} & \begin{matrix}(0)\\ 0\end{matrix} \\[2ex]
\begin{matrix}(-.41-2)\\ .45-2\end{matrix} & \begin{matrix}(.17-4)\\ .20-4\end{matrix} & \begin{matrix}(.32-4)\\ .33-4\end{matrix} & \begin{matrix}(.82-17)\\ 0\end{matrix} & \begin{matrix}(.37-14)\\ 0\end{matrix} & \begin{matrix}(0)\\ 0\end{matrix} \\[2ex]
\begin{matrix}(-.78-2)\\ -.78-2\end{matrix} & \begin{matrix}(.32-4)\\ .33-4\end{matrix} & \begin{matrix}(.61-4)\\ .61-4\end{matrix} & \begin{matrix}(.16-16)\\ 0\end{matrix} & \begin{matrix}(.71-14)\\ 0\end{matrix} & \begin{matrix}(0)\\ 0\end{matrix} \\[2ex]
\begin{matrix}(-.20-14)\\ 0\end{matrix} & \begin{matrix}(.82-17)\\ 0\end{matrix} & \begin{matrix}(.16-16)\\ 0\end{matrix} & \begin{matrix}(.40-29)\\ 0\end{matrix} & \begin{matrix}(.18-26)\\ 0\end{matrix} & \begin{matrix}(0)\\ 0\end{matrix} \\[2ex]
\begin{matrix}(-.91-12)\\ 0\end{matrix} & \begin{matrix}(.37-14)\\ 0\end{matrix} & \begin{matrix}(.71-14)\\ 0\end{matrix} & \begin{matrix}(.18-26)\\ 0\end{matrix} & \begin{matrix}(.83-24)\\ 0\end{matrix} & \begin{matrix}(0)\\ 0\end{matrix} \\[2ex]
\begin{matrix}(0)\\ 0\end{matrix} & \begin{matrix}(0)\\ 0\end{matrix} & \begin{matrix}(0)\\ 0\end{matrix} & \begin{matrix}(0)\\ 0\end{matrix} & \begin{matrix}(0)\\ 0\end{matrix} & \begin{matrix}(0)\\ 0\end{matrix}
\end{bmatrix}$$

$$C_{jk}^{3,R} = \begin{bmatrix}
\begin{matrix}(1.0+0)\\ 1.0+0\end{matrix} & \begin{matrix}(.89-2)\\ .89-2\end{matrix} & \begin{matrix}(-.18-8)\\ 0\end{matrix} & \begin{matrix}(.44-14)\\ 0\end{matrix} & \begin{matrix}(-.91-12)\\ 0\end{matrix} & \begin{matrix}(0)\\ 0\end{matrix} \\[2ex]
\begin{matrix}(.89-2)\\ .89-2\end{matrix} & \begin{matrix}(.79-4)\\ .81-4\end{matrix} & \begin{matrix}(-.16-10)\\ 0\end{matrix} & \begin{matrix}(.39-16)\\ 0\end{matrix} & \begin{matrix}(-.81-4)\\ 0\end{matrix} & \begin{matrix}(0)\\ 0\end{matrix} \\[2ex]
\begin{matrix}(-.18-08)\\ 0\end{matrix} & \begin{matrix}(-.16-10)\\ 0\end{matrix} & \begin{matrix}(.32\ -17)\\ 0\end{matrix} & \begin{matrix}(-.78-23)\\ 0\end{matrix} & \begin{matrix}(.16-20)\\ 0\end{matrix} & \begin{matrix}(0)\\ 0\end{matrix} \\[2ex]
\begin{matrix}(.44-14)\\ 0\end{matrix} & \begin{matrix}(.39-16)\\ 0\end{matrix} & \begin{matrix}(-.80-23)\\ 0\end{matrix} & \begin{matrix}(.19-28)\\ 0\end{matrix} & \begin{matrix}(-.40-26)\\ 0\end{matrix} & \begin{matrix}(0)\\ 0\end{matrix} \\[2ex]
\begin{matrix}(-.91-12)\\ 0\end{matrix} & \begin{matrix}(-.81-14)\\ 0\end{matrix} & \begin{matrix}(.16-20)\\ 0\end{matrix} & \begin{matrix}(-.40-26)\\ 0\end{matrix} & \begin{matrix}(.83-24)\\ 0\end{matrix} & \begin{matrix}(0)\\ 0\end{matrix} \\[2ex]
\begin{matrix}(0)\\ 0\end{matrix} & \begin{matrix}(0)\\ 0\end{matrix} & \begin{matrix}(0)\\ 0\end{matrix} & \begin{matrix}(0)\\ 0\end{matrix} & \begin{matrix}(0)\\ 0\end{matrix} & \begin{matrix}(0)\\ 0\end{matrix}
\end{bmatrix}$$

$$C^{1,\dot{R}}_{jk} = \begin{bmatrix}
\substack{(.34-15)\\ .40-15} & \substack{(.83-13)\\ .90-13} & \substack{(.26-17)\\ .30-17} & \substack{(-.18-7)\\ -.20-7} & \substack{(.75-10)\\ .90-10} & \substack{(-.14-9)\\ -.16-9} \\[2ex]
\substack{(.83-13)\\ .90-13} & \substack{(.20-10)\\ .20-10} & \substack{(.65-15)\\ .71-15} & \substack{(-.45-5)\\ -.45-5} & \substack{(.18-7)\\ .20-7} & \substack{(-.35-7)\\ -.35-7} \\[2ex]
\substack{(.26-17)\\ .30-17} & \substack{(.65-15)\\ .71-15} & \substack{(.21-19)\\ .25-19} & \substack{(-.14-9)\\ -.16-9} & \substack{(.59-12)\\ .70-12} & \substack{(-.11-11)\\ -.12-12} \\[2ex]
\substack{(-.18-7)\\ -.20-7} & \substack{(-.45-5)\\ -.45-5} & \substack{(-.14-9)\\ -.16-9} & \substack{(1.0+0)\\ 1.0+0} & \substack{(-.41-2)\\ -.45-2} & \substack{(.78-2)\\ .78-2} \\[2ex]
\substack{(.75-10)\\ .90-10} & \substack{(.18-7)\\ .20-7} & \substack{(.59-12)\\ .70-12} & \substack{(-.41-2)\\ -.45-2} & \substack{(.17-4)\\ .20-4} & \substack{(-.32-4)\\ -.35-4} \\[2ex]
\substack{(-.14-9)\\ -.16-9} & \substack{(-.35-7)\\ -.35-7} & \substack{(-.11-11)\\ -.12-11} & \substack{(.78-2)\\ .78-2} & \substack{(-.32-4)\\ -.35-4} & \substack{(.61-4)\\ .61-4}
\end{bmatrix}$$

$$C^{2,\dot{R}}_{jk} = \begin{bmatrix}
\substack{(.34-15)\\ .40-16} & \substack{(.83-13)\\ .90-13} & \substack{(-.26-17)\\ -.32-17} & \substack{(-.18-7)\\ -.20-7} & \substack{(.75-10)\\ .90-10} & \substack{(.14-9)\\ .16-9} \\[2ex]
\substack{(.83-13)\\ .90-13} & \substack{(.20-10)\\ .20-10} & \substack{(-.65-15)\\ -.70-15} & \substack{(-.45-5)\\ -.45-5} & \substack{(.18-7)\\ .20-7} & \substack{(.35-7)\\ .35-7} \\[2ex]
\substack{(-.26-17)\\ -.32-17} & \substack{(-.65-15)\\ -.70-15} & \substack{(.21-19)\\ .25-19} & \substack{(.14-9)\\ .16-9} & \substack{(-.59-12)\\ -.70-12} & \substack{(-.11-11)\\ -.12-11} \\[2ex]
\substack{(-.18-7)\\ -.20-7} & \substack{(-.45-5)\\ -.45-5} & \substack{(.14-9)\\ .16-9} & \substack{(1.0+0)\\ 1.0+0} & \substack{(-.41-2)\\ -.45-2} & \substack{(-.78-2)\\ -.78-2} \\[2ex]
\substack{(.75-10)\\ .90-10} & \substack{(.18-7)\\ .20-7} & \substack{(-.59-12)\\ -.70-12} & \substack{(-.41-2)\\ -.45-2} & \substack{(.17-4)\\ .20-4} & \substack{(.32-4)\\ .35-4} \\[2ex]
\substack{(.14-9)\\ .16-9} & \substack{(.35-7)\\ .35-7} & \substack{(-.11-11)\\ -.12-11} & \substack{(-.78-2)\\ -.78-2} & \substack{(.32-4)\\ .35-4} & \substack{(.61-4)\\ .61-4}
\end{bmatrix}$$

$$
C^{3,\dot{R}}_{jk} = \begin{bmatrix}
(.16\text{-}14) & (-.18\text{-}12) & (-.28\text{-}23) & (.40\text{-}7) & (.35\text{-}9) & (-.71\text{-}16) \\
.16\text{-}14 & -.18\text{-}12 & 0 & .40\text{-}7 & .35\text{-}9 & 0 \\
(-.18\text{-}12) & (.20\text{-}10) & (-.32\text{-}21) & (-.45\text{-}5) & (-.40\text{-}7) & (.80\text{-}14) \\
-.18\text{-}12 & .20\text{-}10 & 0 & -.45\text{-}5 & -.40\text{-}7 & 0 \\
(-.28\text{-}23) & (.32\text{-}2) & (0) & (-.71\text{-}16) & (-.63\text{-}18) & (.13\text{-}24) \\
0 & 0 & 0 & 0 & 0 & 0 \\
(.40\text{-}7) & (-.45\text{-}5) & (-.71\text{-}16) & (1.0+0) & (.89\text{-}2) & (-.18\text{-}8) \\
.40\text{-}7 & -.45\text{-}5 & 0 & 1.0+0 & .90\text{-}2 & 0 \\
(.35\text{-}9) & (-.40\text{-}7) & (-.63\text{-}18) & (.89\text{-}2) & (.79\text{-}4) & (-.16\text{-}10) \\
.35\text{-}9 & -.40\text{-}7 & 0 & .90\text{-}2 & .80\text{-}4 & 0 \\
(-.71\text{-}16) & (.80\text{-}14) & (.13\text{-}24) & (-.18\text{-}8) & (-.16\text{-}10) & (.32\text{-}17) \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

In the above, the notation a + b stands for a x 10^b.

In addition, several values of the Jacobian matrix $\partial x_i(t)/\partial a_j$ were checked for t equal to one-half period and were found to be correct. Further, the covariance matrix obtained using all the $C^{i,R}$ and $C^{i,\dot{R}}$ with σ_R = 15 m and $\sigma_{\dot{R}}$ = 3 cm/sec. were checked for reasonableness and found to be very nearly equal to the estimated values.

Computations Without Boost

The following computations have been performed, and the results are presented graphically. In all these cases the parameters used are:

$a_1 = 1.7525 \times 10^6$ m

$a_4 = 0$ m/sec

$a_5 = 1.70172 \times 10^3$ m/sec

$\xi = 0^\circ$

$\eta = 180^\circ$

$\zeta = 180^\circ$

$\omega_e = .7291160 \times 10^{-4}$ rad/sec

$\omega_m = .42360 \times 10^{-6}$ rad/sec

$\rho_e = 6.3781 \times 10^6$ m

$\rho_m = 3.85 \times 10^8$ m

$\mu = 4.896 \times 10^{12}$ m^3/sec^2

$L = 15^\circ$

$\ell = 90^\circ$

These conditions correspond to a vehicle at perilune of a Hohmann transfer from 8 n.mi. altitude to 80 n.mi. altitude, the nominal lunar rendezvous maneuver. Perilune is on the Earth-Moon line, and the orbit of the vehicle is in the plane of Earth-Moon rotation.

The orbit period is 116.2 minutes. Half a period is 58.09 minutes. Through a minor error, the time used for prediction of errors at time of nominal rendezvous was 59.08 minutes. This has a negligible effect upon the results of the computations since error in position and not position is calculated. Error in position is a slowly varying quantity, while position of the LEM or position of the LEM relative to the CM/SM is not. The error was discovered early in the computations, but because of its very small effect and because of the desire to obtain results in time for the final report, it was not corrected.

Three stations are used

Station	λ (latitude)	α (longitude)
Madrid	41°	-4°
Johannesburg	-26°	28°
Woomera	-30°	138°

Information matrices for range and range-rate from these stations have been computed for 1, 4, 9, 16 and 25 successive 1 minute observations after the initial conditions. The Jacobian matrix for computing the errors at rendezvous was computed for 59.08 minutes as described above.

Using $\sigma_R = 15$ m and $\sigma_{\dot{R}} = 3$ cm/sec for one minute observations, the covariance matrices of errors at 59.08 minutes have been computed for the following conditions.

Stations			Data			Total Observation Time (minutes)					Remarks
M	J	W	R	$\dot{R}$	A priori	1	4	9	16	25	
x	x	x	x	x		x	x	x	x	x	
			x	x	x	x	x	x	x	x	Note 1
			x				x	x	x	x	Note 2
			x		x	x	x	x	x	x	Note 1
				x			x	x	x	x	Note 2
				x	x	x	x	x	x	x	Note 1
x	x		x	x			x	x	x	x	Note 2
			x	x	x	x	x	x	x	x	Note 1
			x				x	x	x	x	Note 2
			x		x	x	x	x	x	x	Note 1
				x			x	x	x	x	Note 2
				x	x	x	x	x	x	x	Note 1
x			x	x			x	x	x	x	Note 2
			x	x	x	x	x	x	x	x	Note 1
			x					x	x	x	Note 2
			x		x	x	x	x	x	x	Note 1
				x				x	x	x	Note 2
				x	x	x	x	x	x	x	Note 1
				x	x	x	x	x	x	x	Note 3
				x	x	x	x	x	x	x	Note 4

Note 1:

A priori data

$$\sigma_{a_1} = \sigma_{a_2} = \sigma_{a_3} = 10^3 \text{m}$$

$$\sigma_{a_4} = \sigma_{a_5} = \sigma_{a_6} = \sqrt{10} \text{ m/sec.}$$

Note 2:

The number of minutes of observations to be able to determine a_1 through a_6 depends on the number of stations and whether range, range-rate or both are used. The total number of independent observations must be at least 6. For this reason, the covariance matrix of the errors at rendezvous starts with other than 1 minute of observations in some instances.

Note 3:

A priori data

$$\sigma_{a_1} = \sigma_{a_2} = \sigma_{a_3} = \sqrt{10} \quad 10^3 \text{ m}$$

$$\sigma_{a_4} = \sigma_{a_5} = \sigma_{a_6} = \sqrt{10} \quad 10 \text{ m/sec.}$$

Note 4:

A priori data

$$\sigma_{a_1} = \sigma_{a_2} = \sigma_{a_3} = 10^4 \text{ m}$$

$$\sigma_{a_5} = \sigma_{a_6} = \sigma_{a_7} = \sqrt{10} \quad 10^2 \text{ m/sec.}$$

On the basis of these error covariance matrices σ_x, σ_y and σ_z have been plotted, as has $(\sigma_x^2 + \sigma_y^2 + \sigma_z^2)^{1/2}$ which is referred to as the RMS error at 59.08 minutes.

Further, a major portion of the error is in the y direction (along the direction of motion). Now, the CM/SM is in a circular orbit, and the LEM is near apolune in a near-circular orbit. Thus, the relative motion is in the y direction, and for small errors in velocity is approximately 29.7 m/sec. As a consequence, a short time before or after the nominal rendezvous time the error in the y direction becomes zero while the errors in the x and z directions remain essentially unchanged. The RMS time between nominal rendezvous and this time of minimum miss is $\sigma_y/29.7$ seconds and has been plotted for those cases in which the velocity errors are small. The corresponding minimum RMS misses have also been plotted.

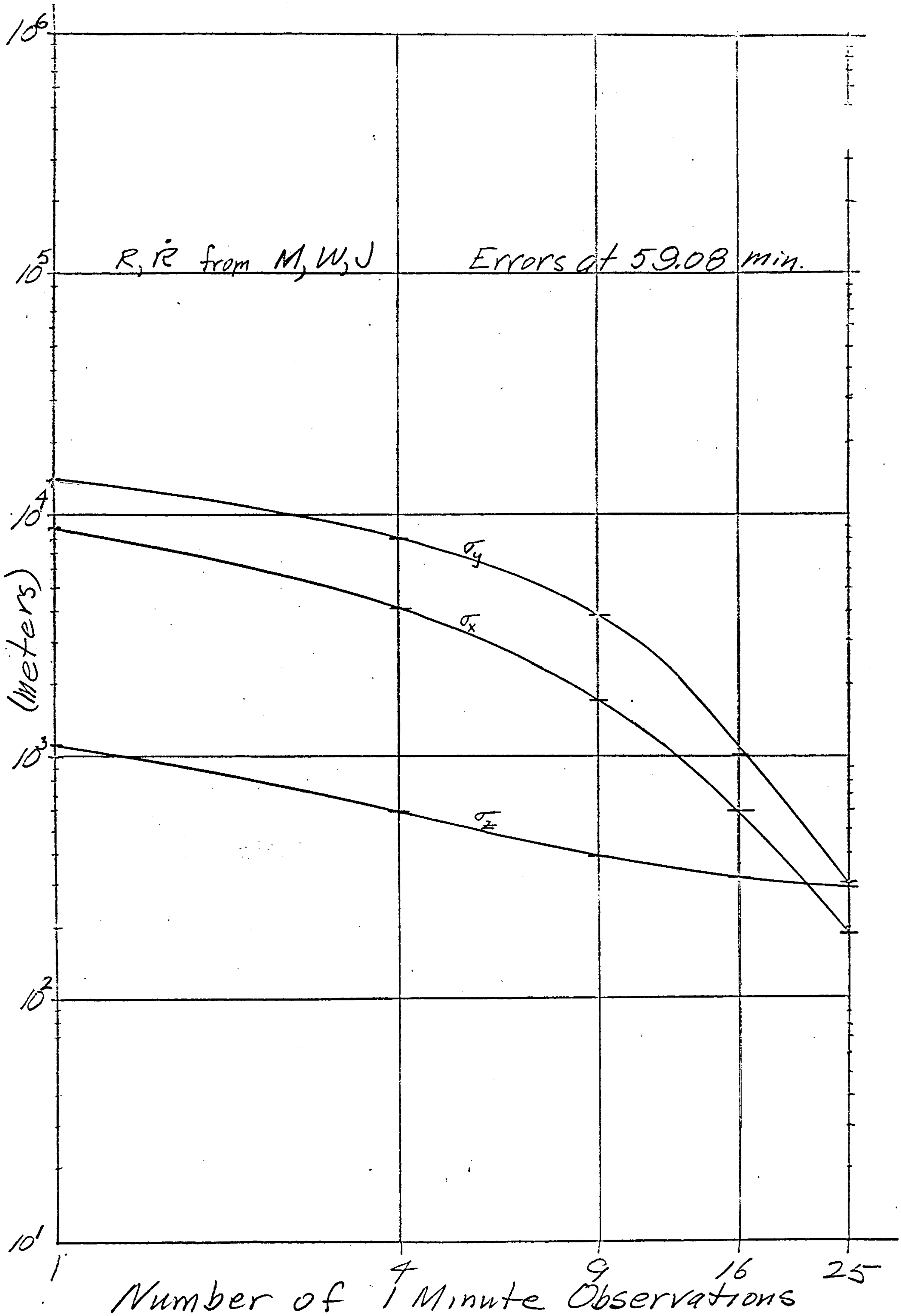
Computations With Boost

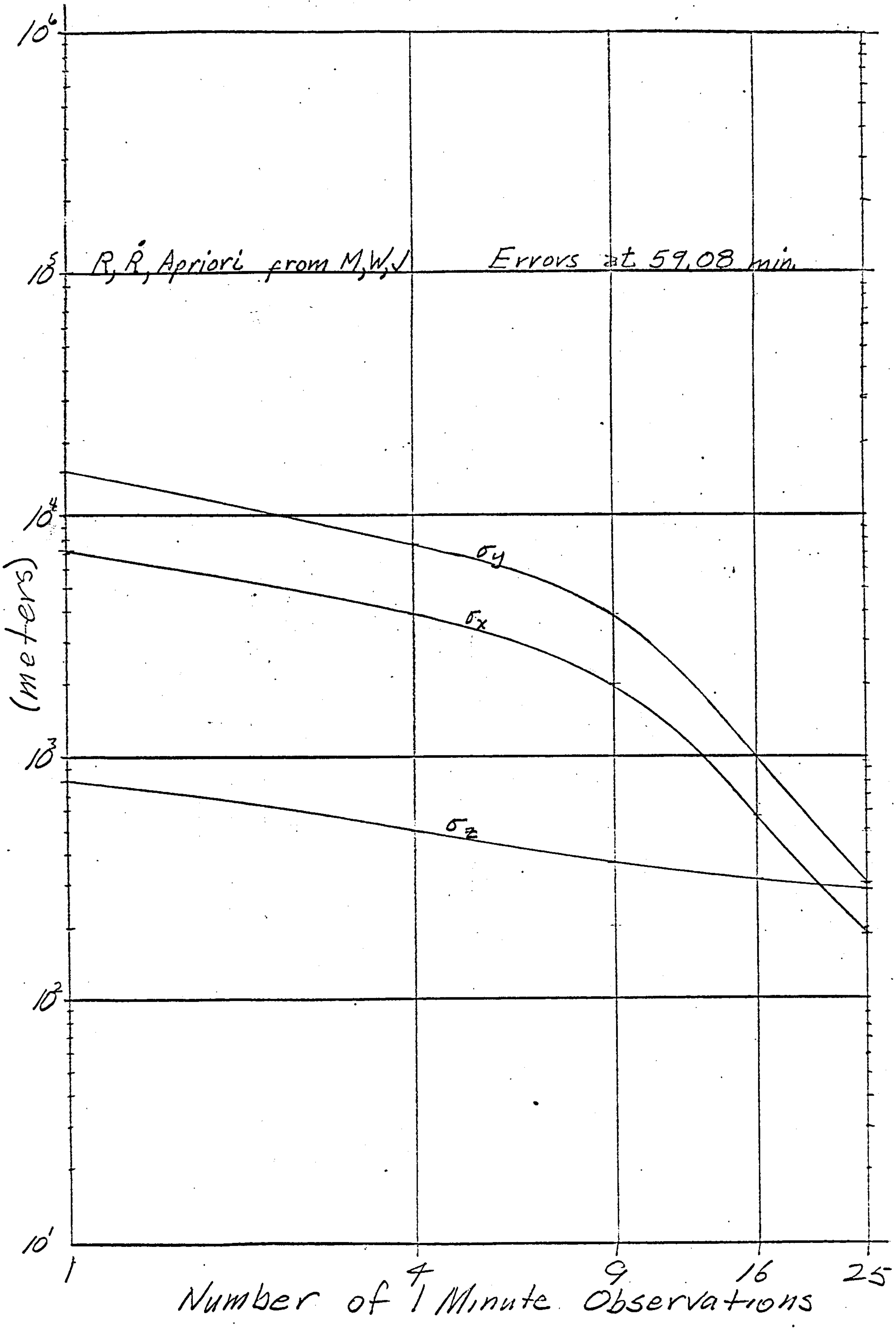
Additional information matrices corresponding to range and range-rate measurements from Madrid, Johannesburg and Woomera at one minute intervals from 17 to 25 minutes has been computed; so has the Jacobian matrix $\partial x_i / \partial a_j$ corresponding to 16 minutes. With these additional matrices, the following problem has been solved, according to the method outlined in Apollo Note No. 95.

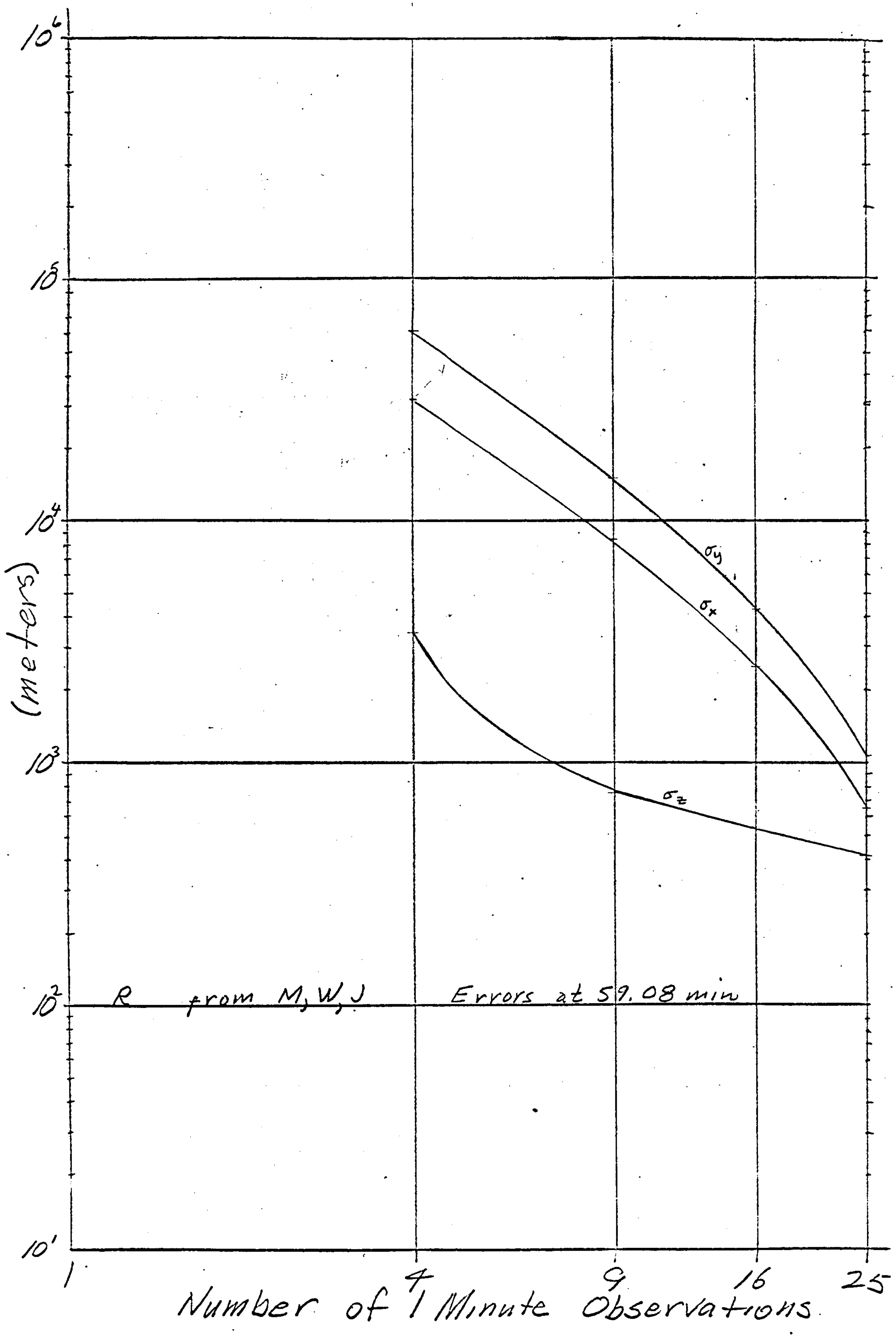
The vehicle has the orbit parameters described under "Computations Without Boost," above. The vehicle is observed in range and range-rate from Madrid, Woomera, and Johannesburg for 16 minutes. An orbit correcting boost is then made with RMS errors of 1.0 m/sec. in the x, y and z directions. The actual boost is assumed zero; this does not substantially affect the errors in estimated position at future time. Then the vehicle is observed for another 9 minutes, and the error covariance matrix at rendezvous computed.

From this the RMS errors at 59.08 minutes (1250m), the RMS difference between the nominal time of rendezvous and the time of minimum RMS miss (32.4 sec) and the minimum RMS miss (790m) are computed.

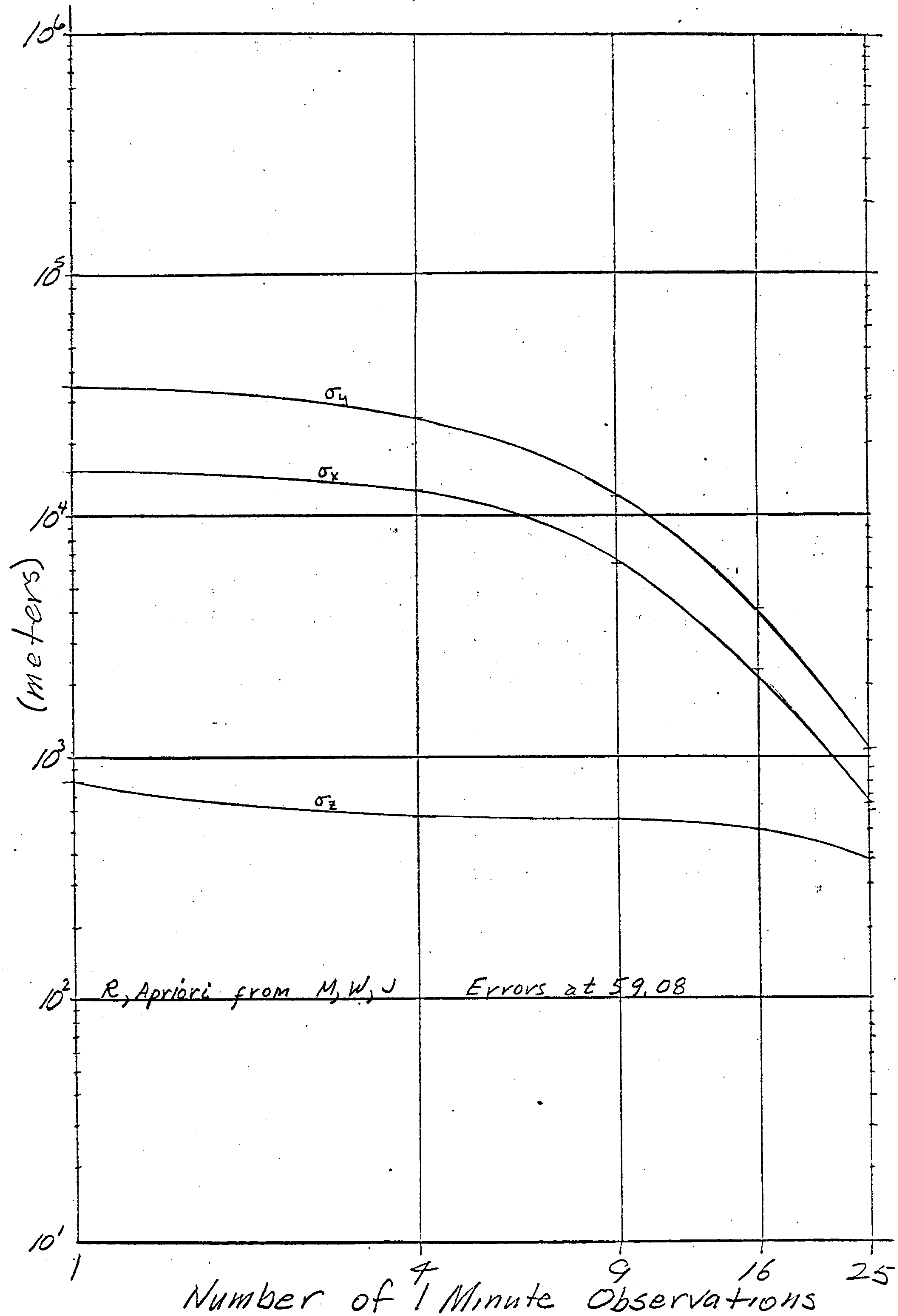
In performing the calculations for determining the errors with boost, it has been necessary to "fool" the program to obtain the desired results since the modifications to make these computations simple and routine have not yet been completed. At present these computations require several computer passes performed at least a half day apart, with a consequent lengthy throughput time.


R, Ṙ from M, W, J
Errors at 59.08 min.
(meters)
10^6
10^5
10^4
10^3
10^2
10^1
σ_y
σ_x
σ_z
1
4
9
16
25
Number of 1 Minute Observations

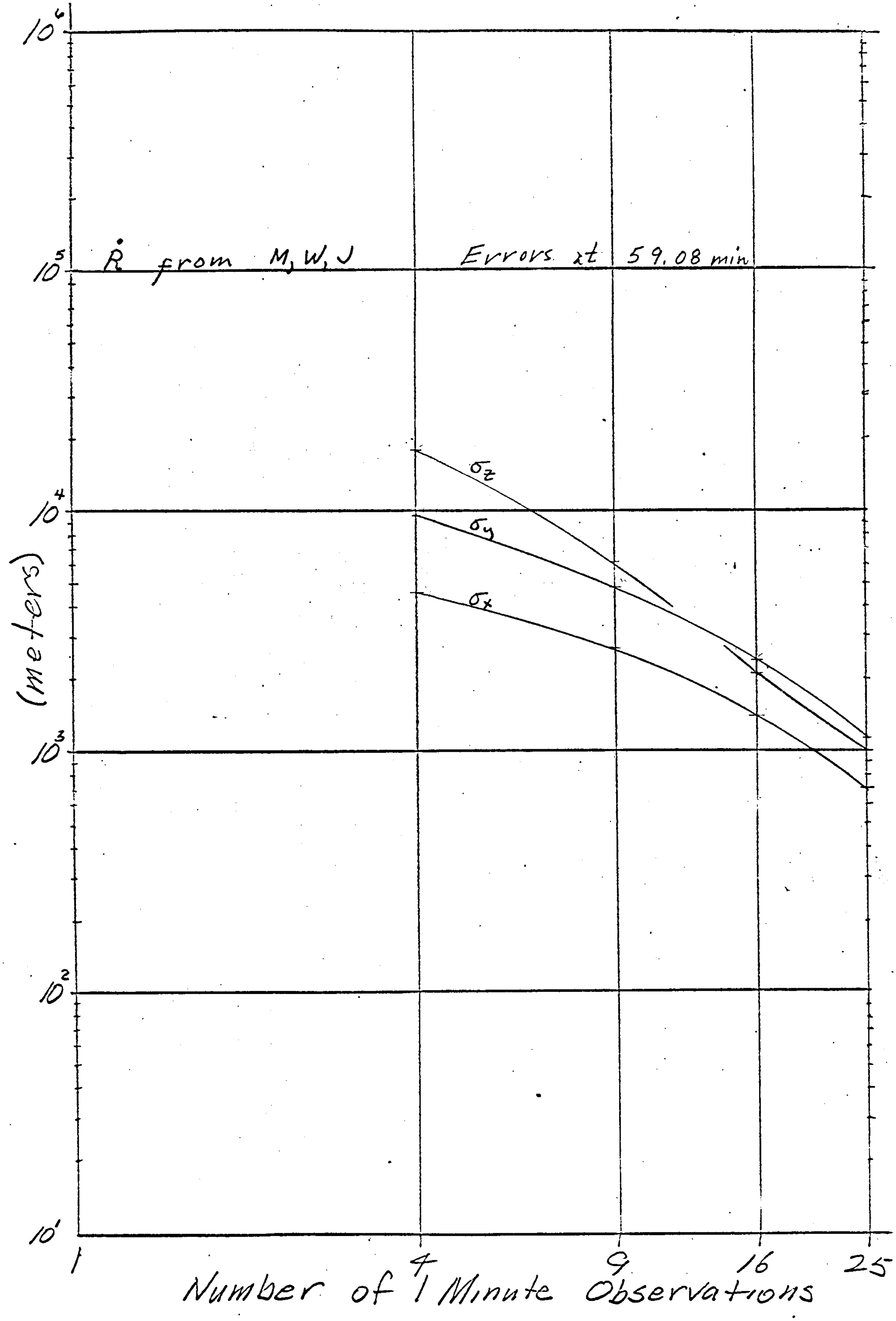

R, Ṙ, Apriori from M,W,J
Errors at 59.08 min.
10^6
10^5
10^4
10^3
10^2
10^1
(meters)
σ_y
σ_x
σ_z
1
4
9
16
25
Number of 1 Minute Observations



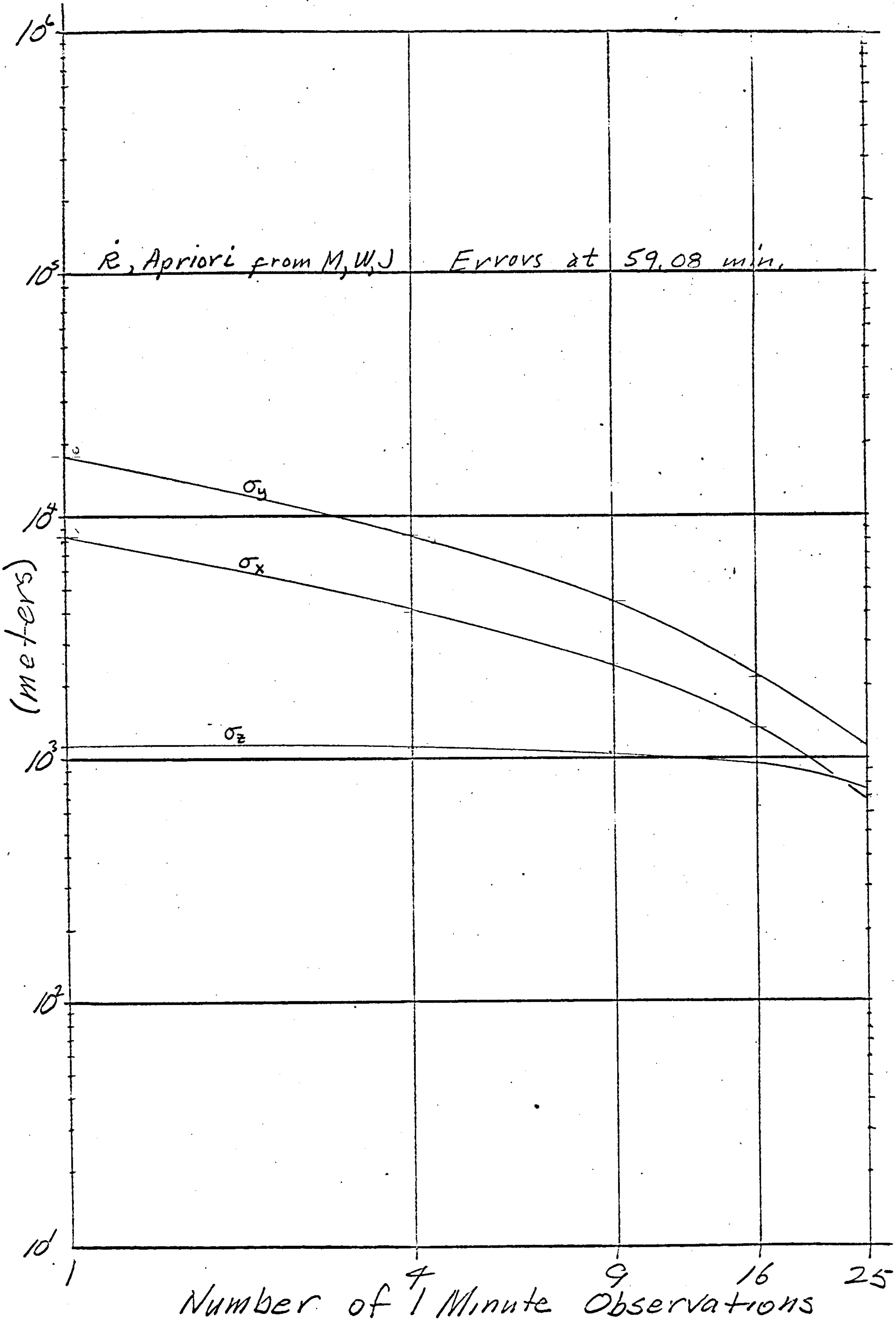
10^6
10^5
10^4
10^3
10^2
10^1
(meters)
σ_y
σ_x
σ_z
R from M, W, J
Errors at 59.08 min
1
4
9
16
25
Number of 1 Minute Observations



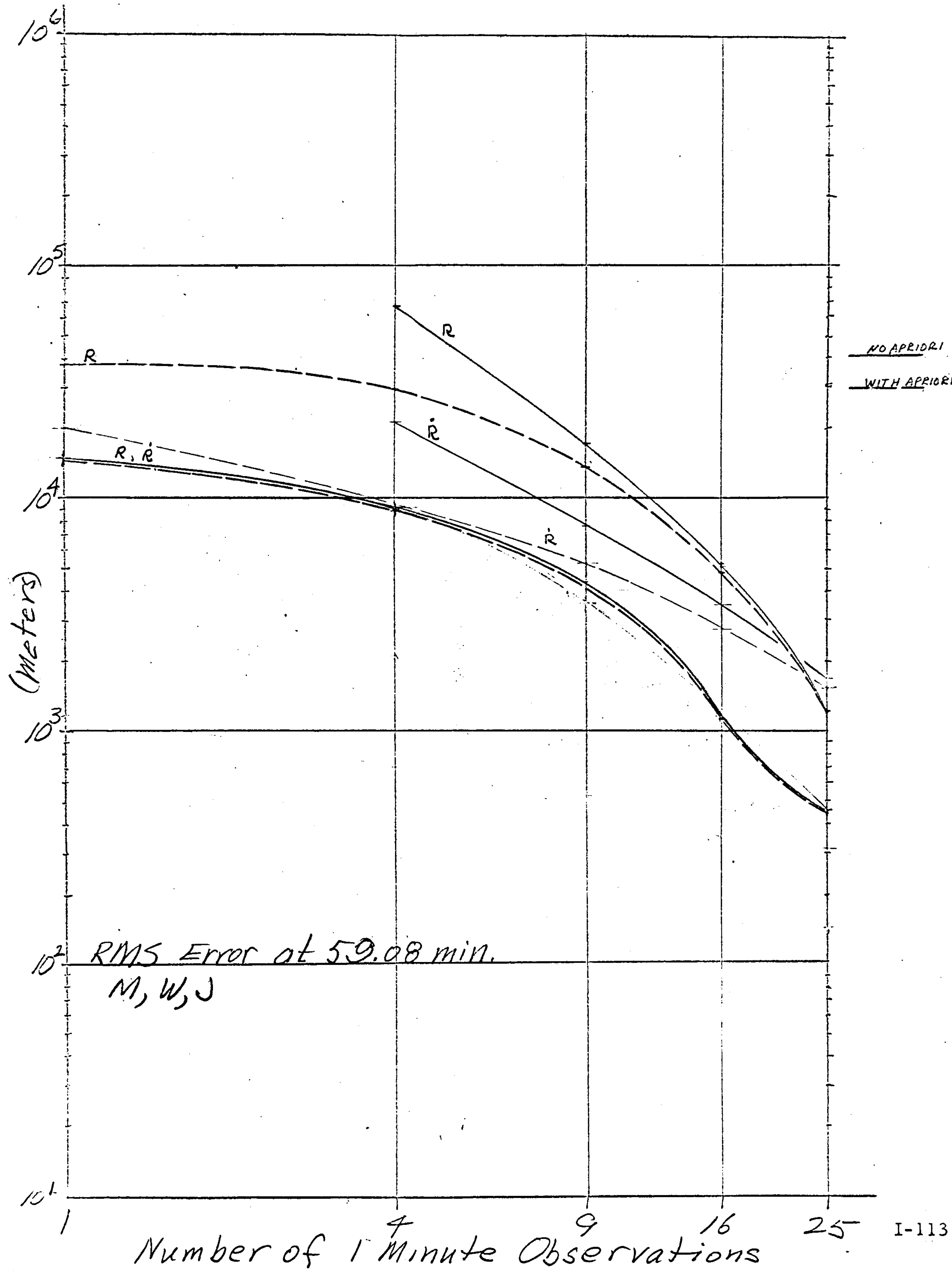
10^6
10^5
10^4
10^3
10^2
10^1
(meters)
σ_y
σ_x
σ_z
R, Apriori from M, W, J
Errors at 59.08
1
4
9
16
25
Number of 1 Minute Observations



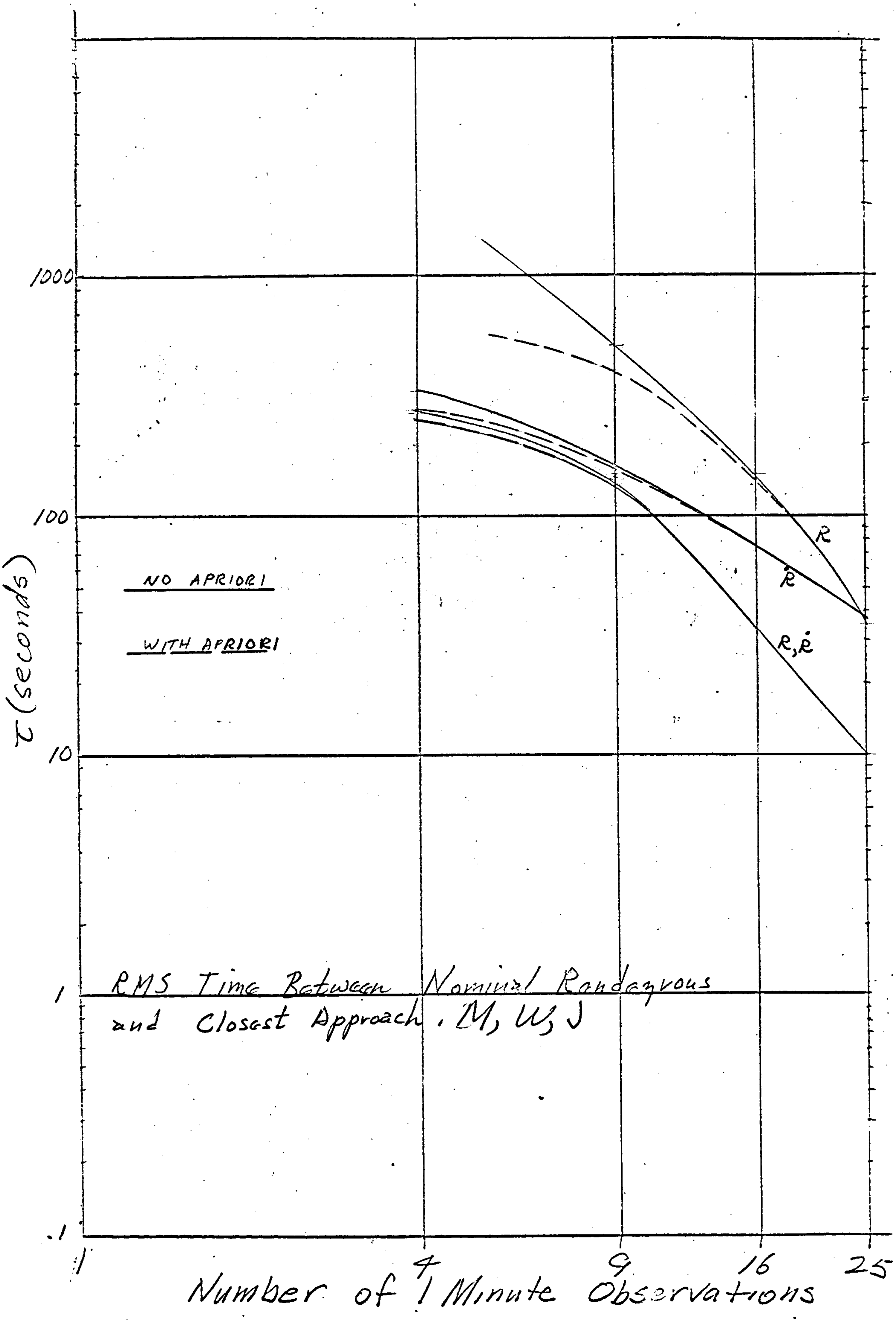
$\dot{R}$ from M, W, J
Errors at 59.08 min
10^6
10^5
10^4
10^3
10^2
10^1
(meters)
σ_z
σ_y
σ_x
1
4
9
16
25
Number of 1 Minute Observations

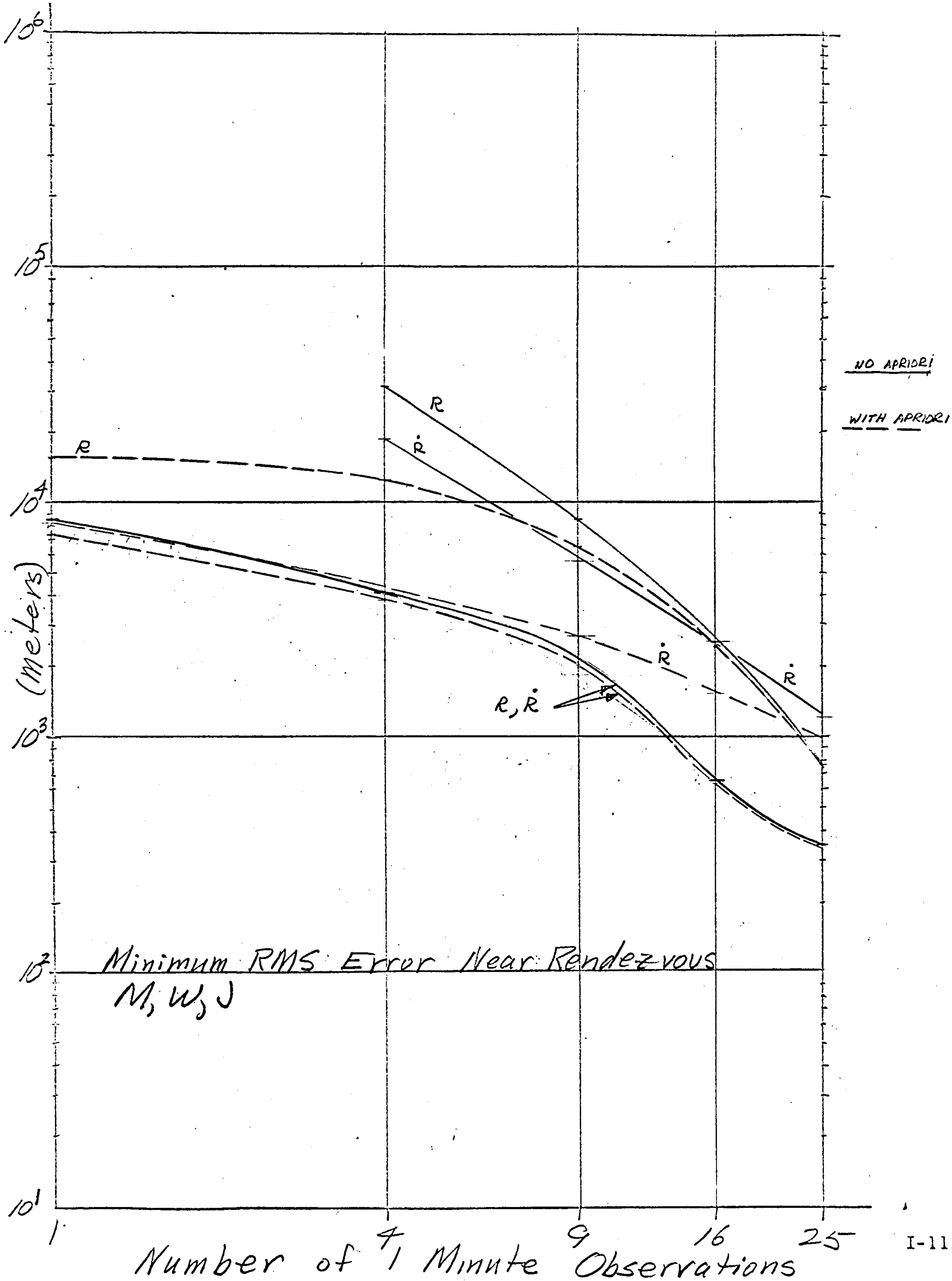


$\dot{R}$, Apriori from M, W, J   Errors at 59.08 min.
10^6
10^5
10^4
10^3
10^2
10^1
(meters)
σ_y
σ_x
σ_z
1
4
9
16
25
Number of 1 Minute Observations

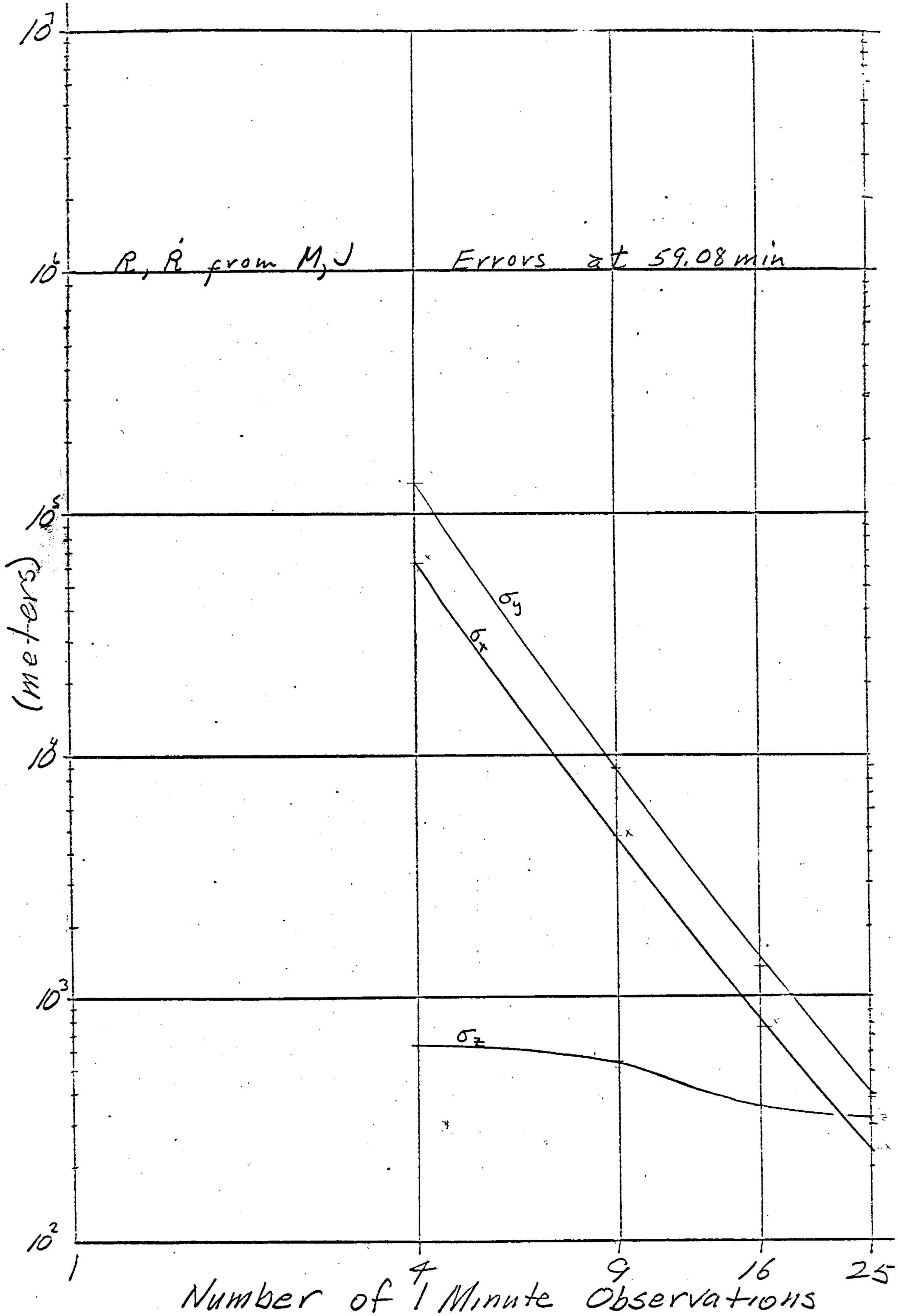


10^6
10^5
10^4
10^3
10^2
10^1
R
R
$\dot{R}$
R, $\dot{R}$
$\dot{R}$
NO APRIORI
WITH APRIORI
(Meters)
RMS Error at 59.08 min.
M, W, J
1
4
9
16
25
Number of 1 Minute Observations

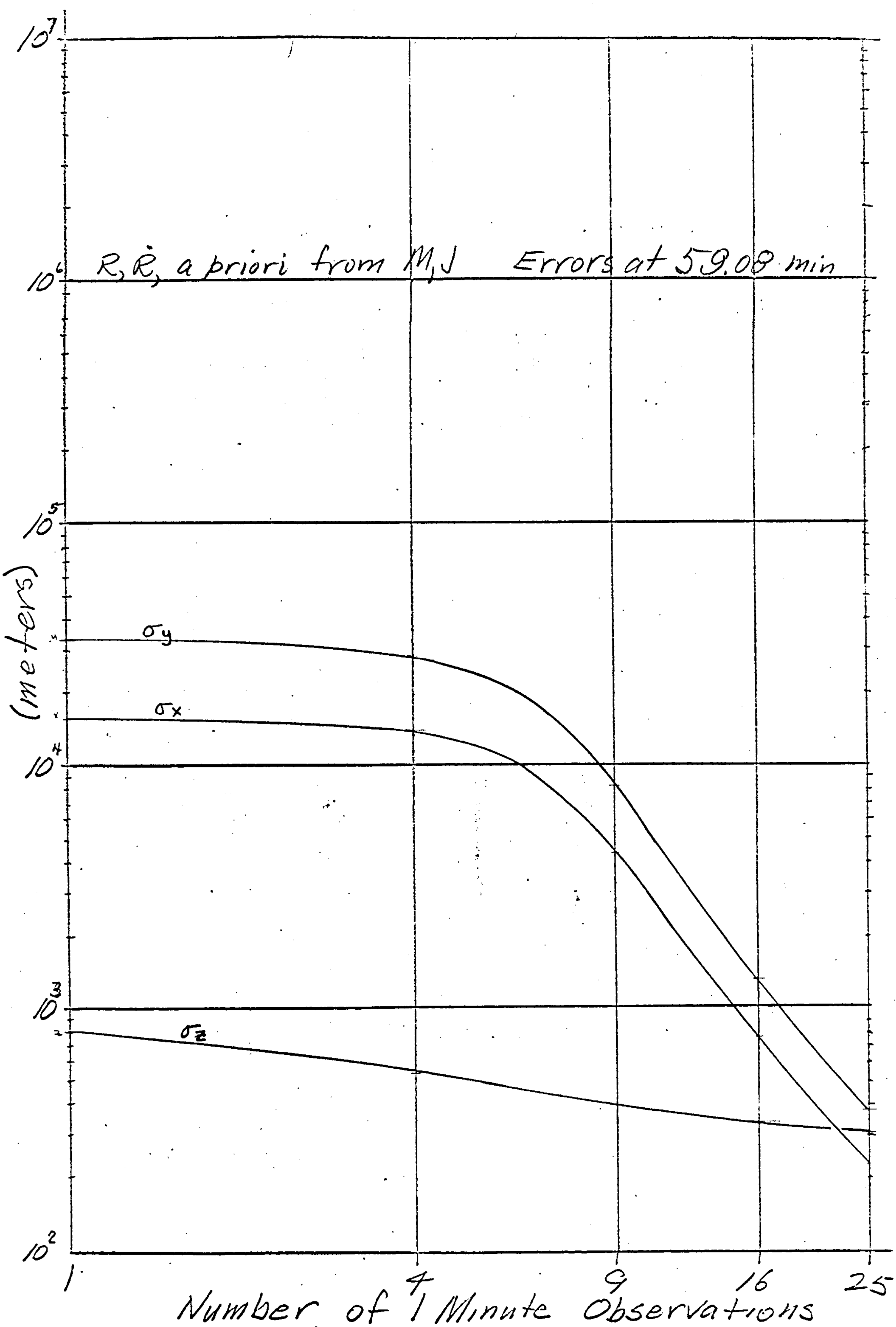

1000
100
10
1
.1
τ (seconds)
NO APRIORI
WITH APRIORI
R
$\dot{R}$
R, $\dot{R}$
RMS Time Between Nominal Rendezvous and Closest Approach. M, W, J
1
4
9
16
25
Number of 1 Minute Observations



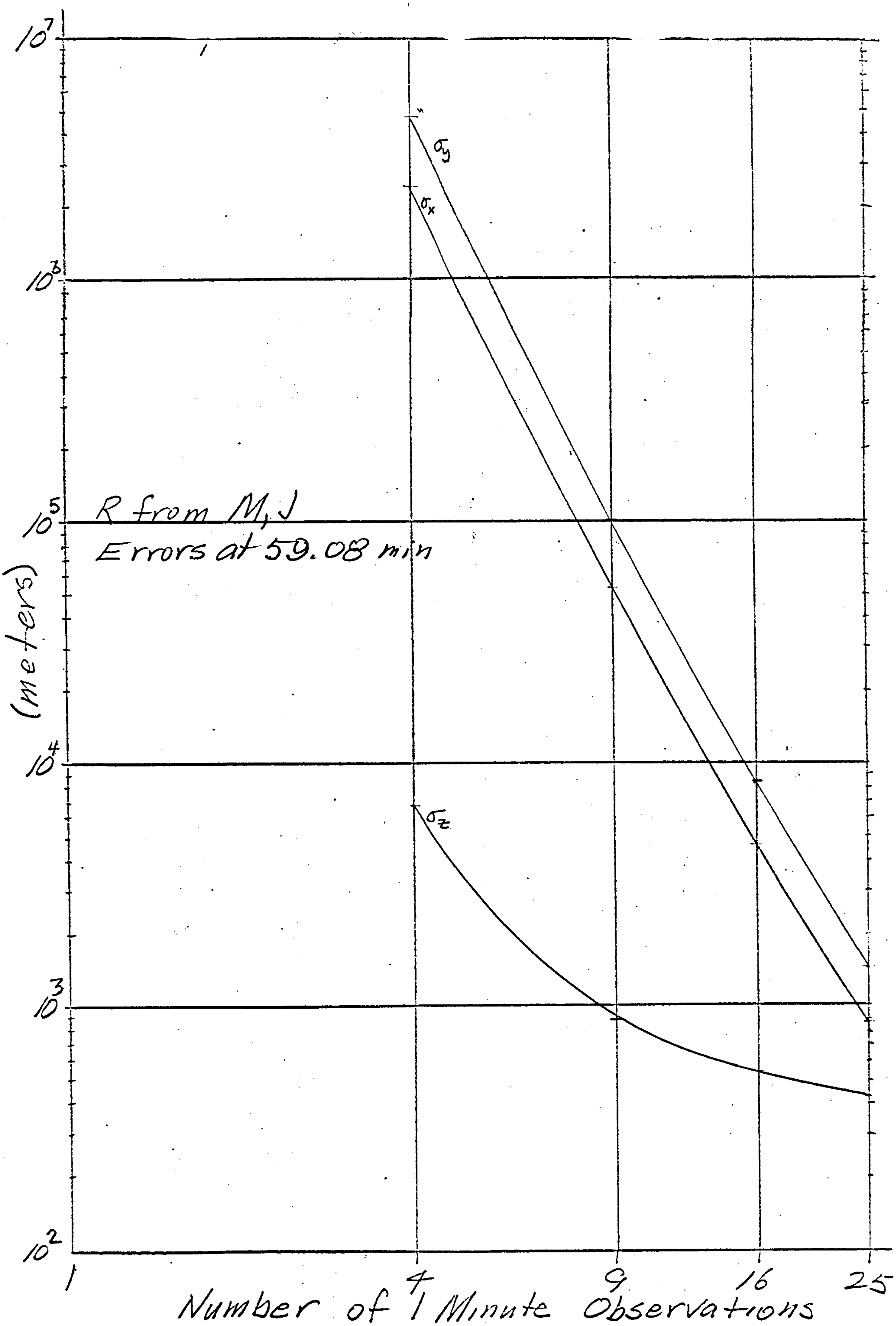
10^6
10^5
10^4
10^3
10^2
10^1
(meters)
NO APRIORI
WITH APRIORI
R
Ṙ
R
Ṙ
Ṙ
R, Ṙ
Minimum RMS Error Near Rendezvous
M, W, J
1
4
9
16
25
Number of 1 Minute Observations



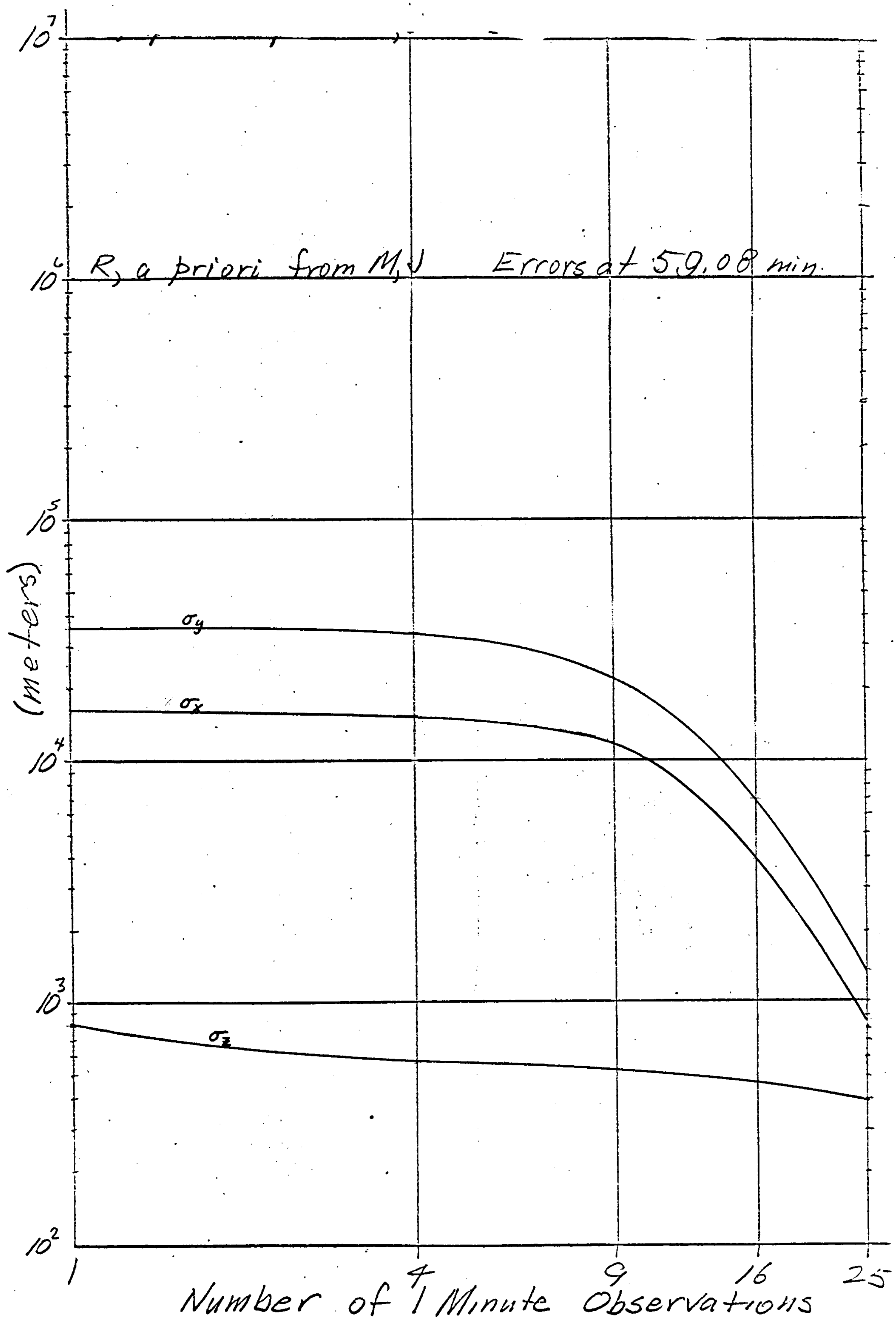
R, Ṙ from M, J
Errors at 59.08 min
(meters)
10^7
10^6
10^5
10^4
10^3
10^2
σ_y
σ_x
σ_z
1
4
9
16
25
Number of 1 Minute Observations

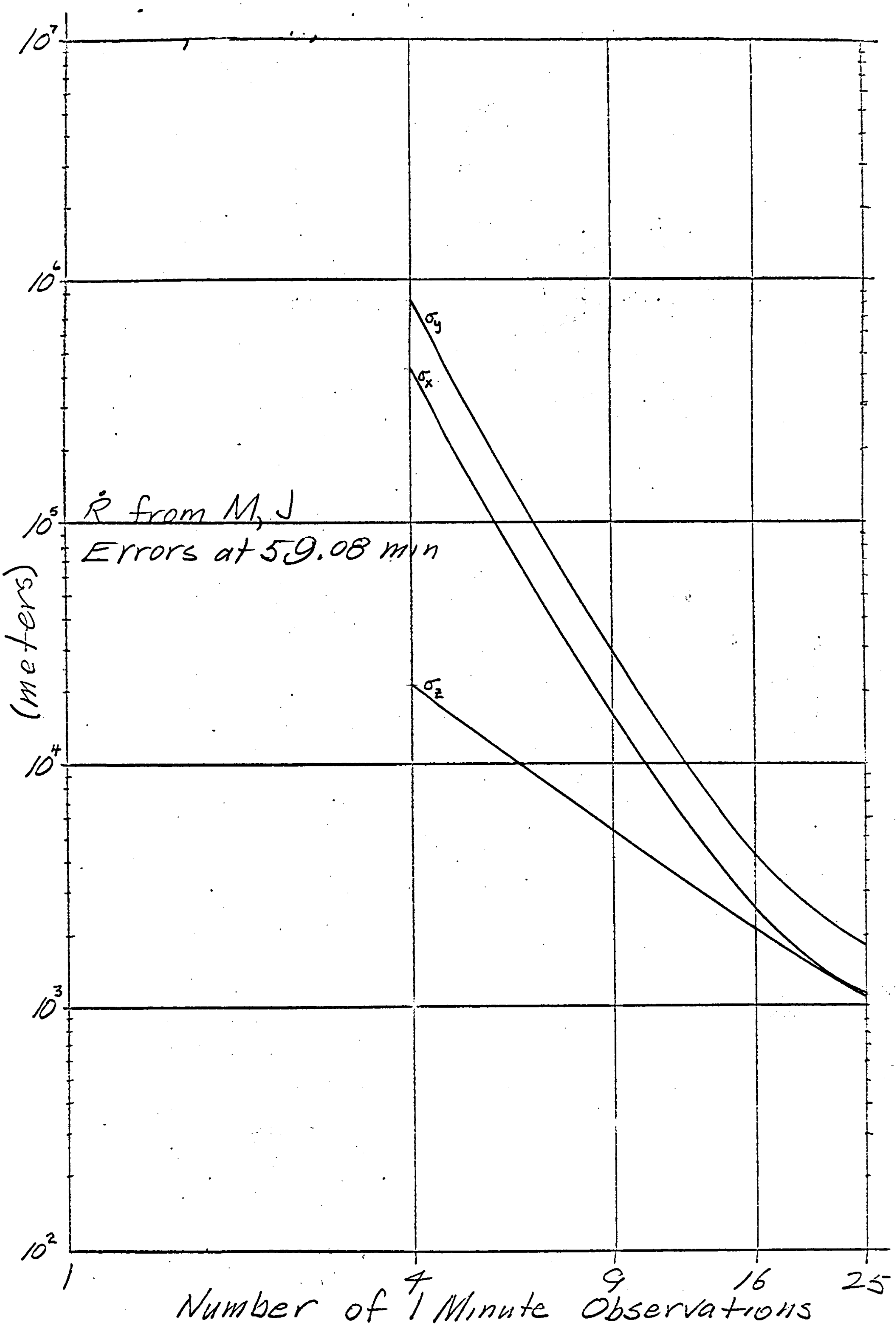


R, Ṙ, a priori from M, J
Errors at 59.08 min
10^7
10^6
10^5
10^4
10^3
10^2
(meters)
σ_y
σ_x
σ_z
1
4
9
16
25
Number of 1 Minute Observations

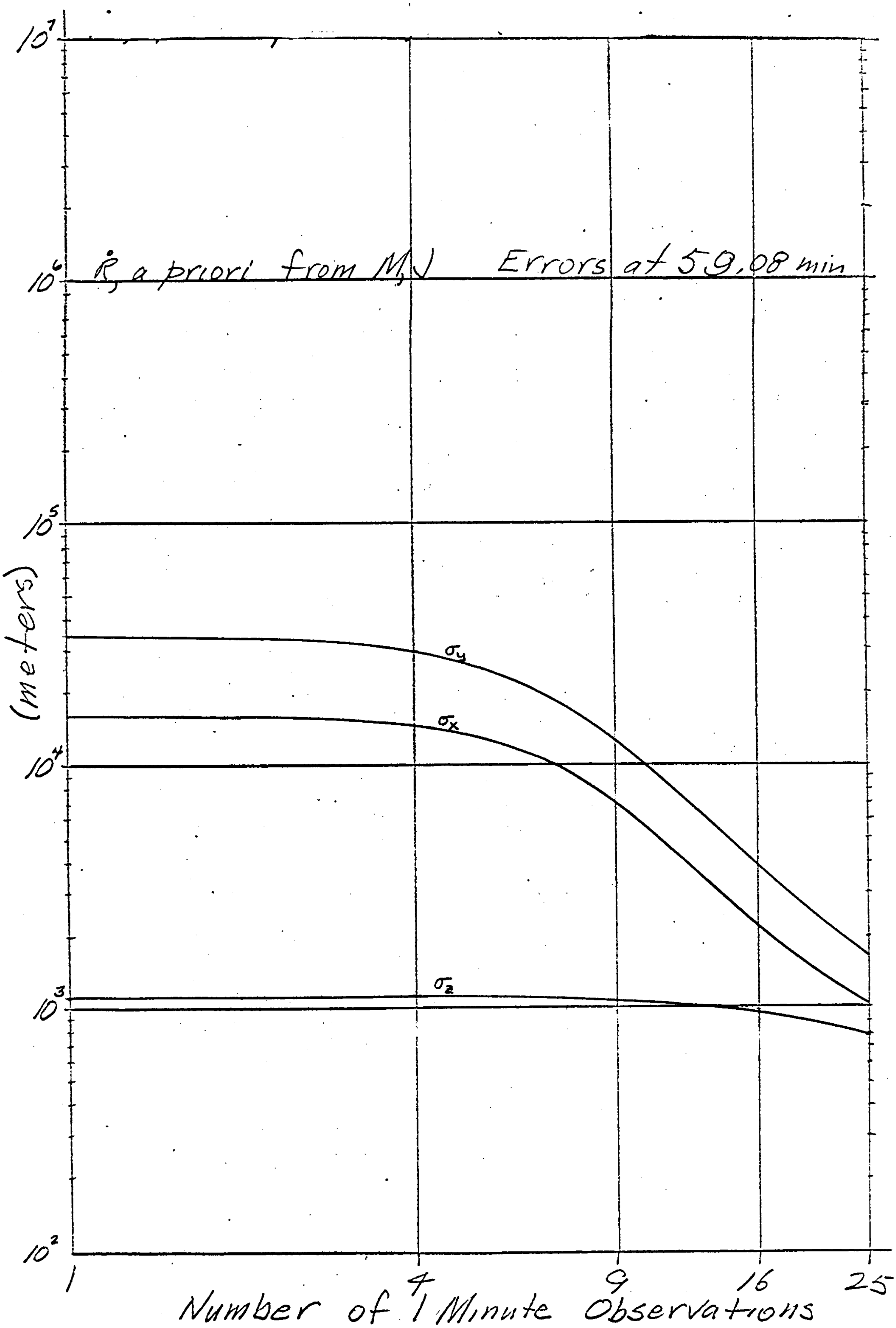


10^7
10^6
10^5
10^4
10^3
10^2
(meters)
σ_y
σ_x
σ_z
R from M, J
Errors at 59.08 min
1
4
9
16
25
Number of 1 Minute Observations

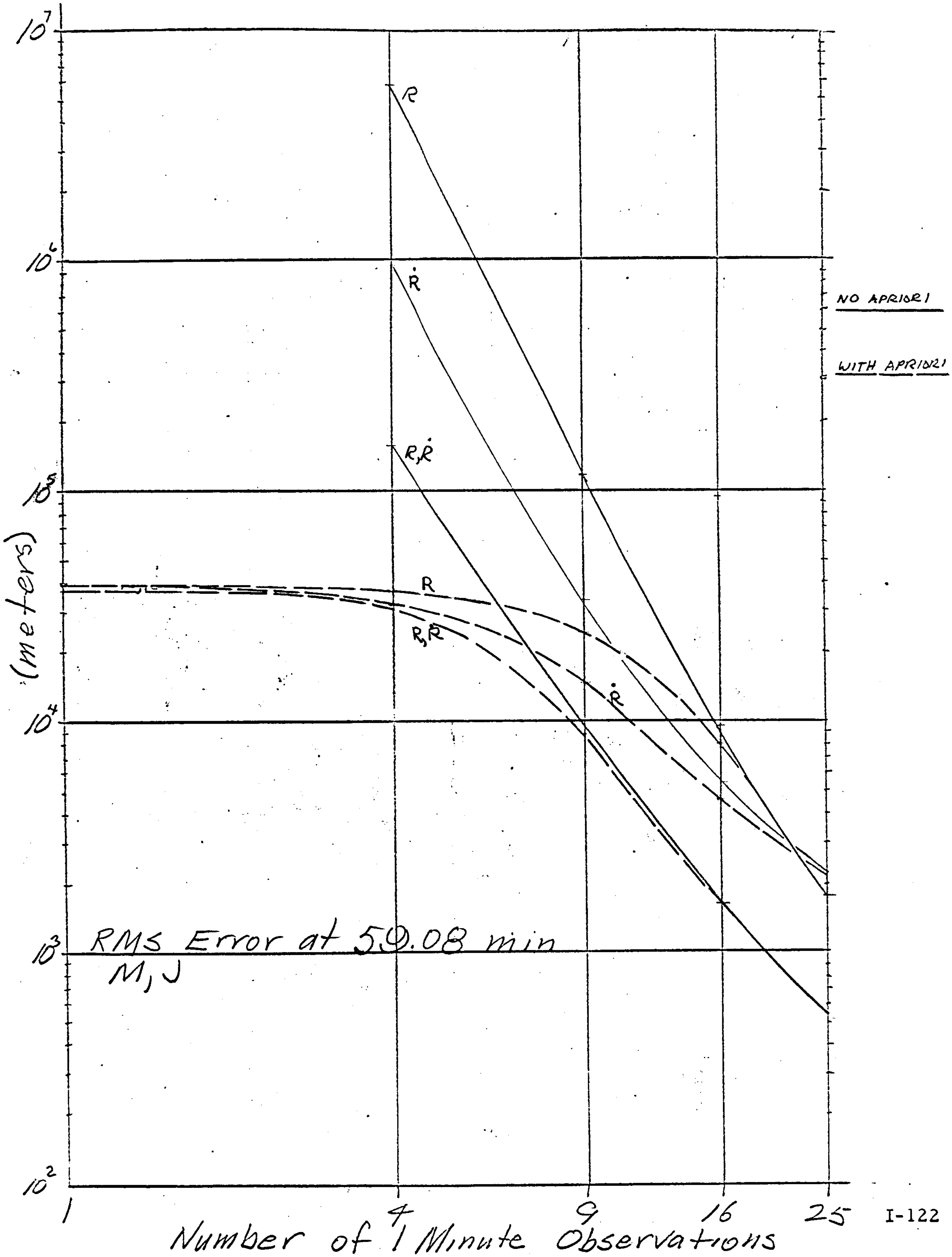

R, a priori from M, J
Errors at 59.08 min.
(meters)
10^7
10^6
10^5
10^4
10^3
10^2
σ_y
σ_x
σ_z
1
4
9
16
25
Number of 1 Minute Observations



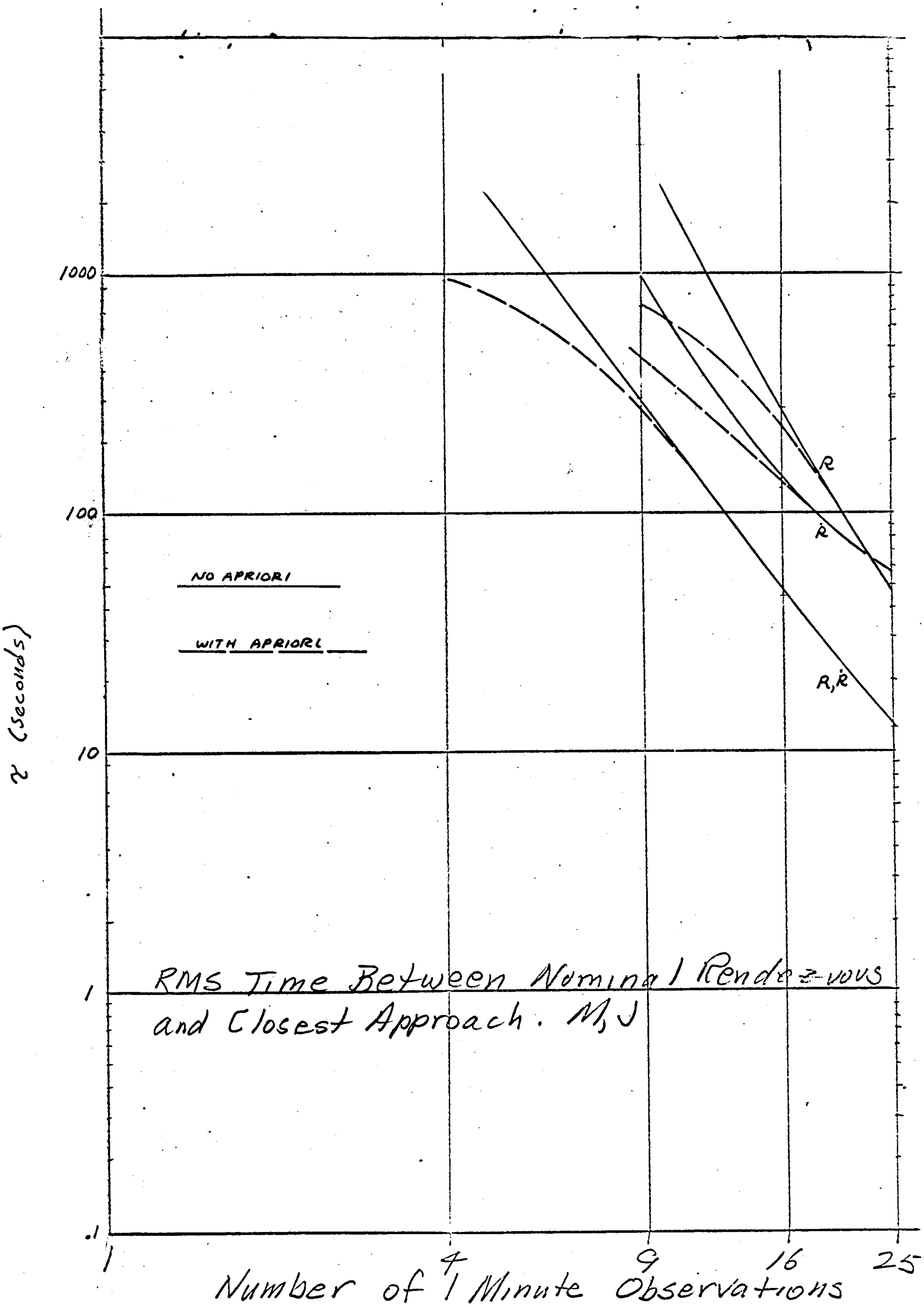
10^7
10^6
10^5
10^4
10^3
10^2
$\dot{R}$ from M, J
Errors at 59.08 min
(meters)
σ_y
σ_x
σ_z
1
4
9
16
25
Number of 1 Minute Observations

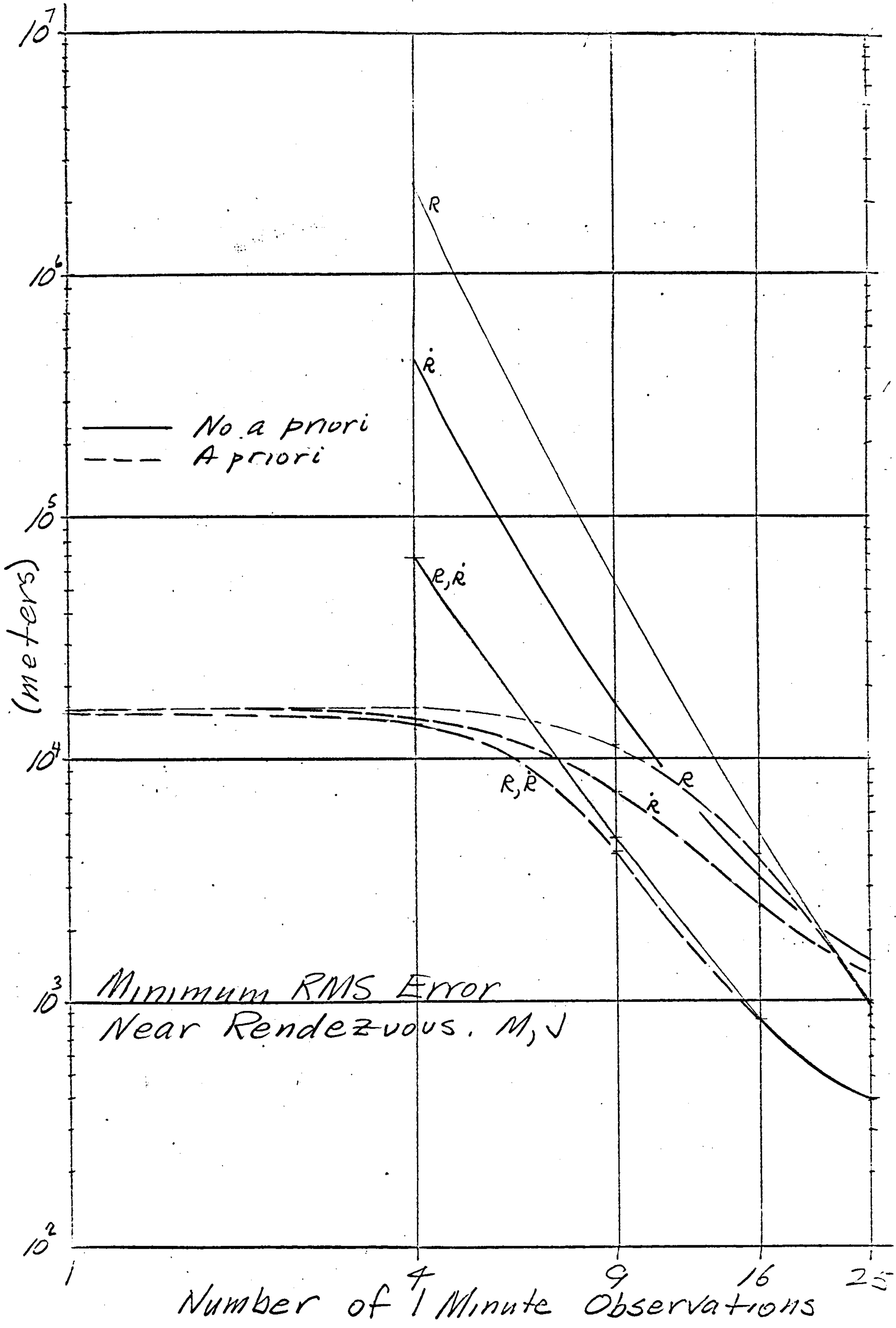


$\dot{R}$, a priori from M, J
Errors at 59.08 min
10^7
10^6
10^5
10^4
10^3
10^2
(meters)
σ_y
σ_x
σ_z
1
4
9
16
25
Number of 1 Minute Observations

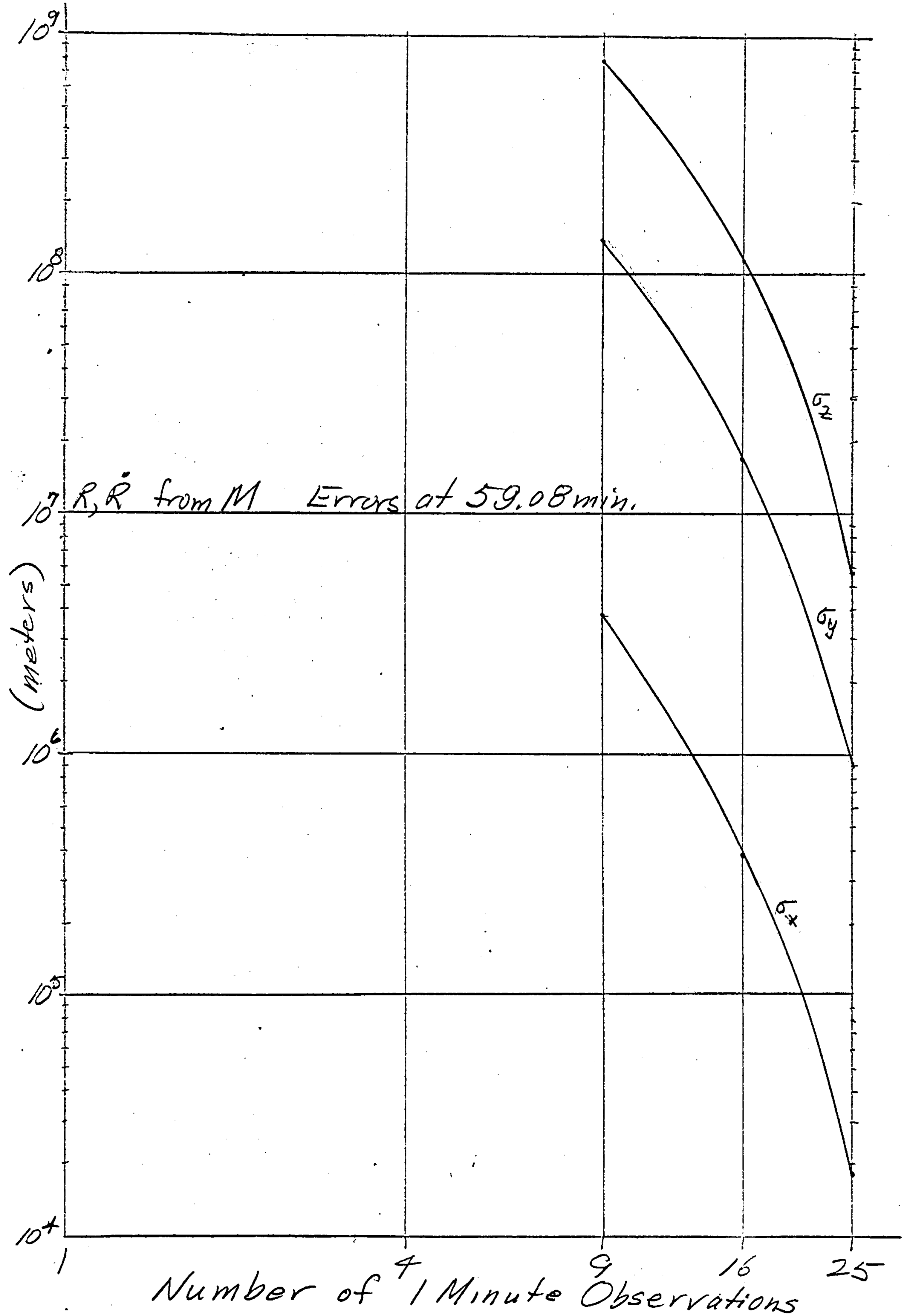


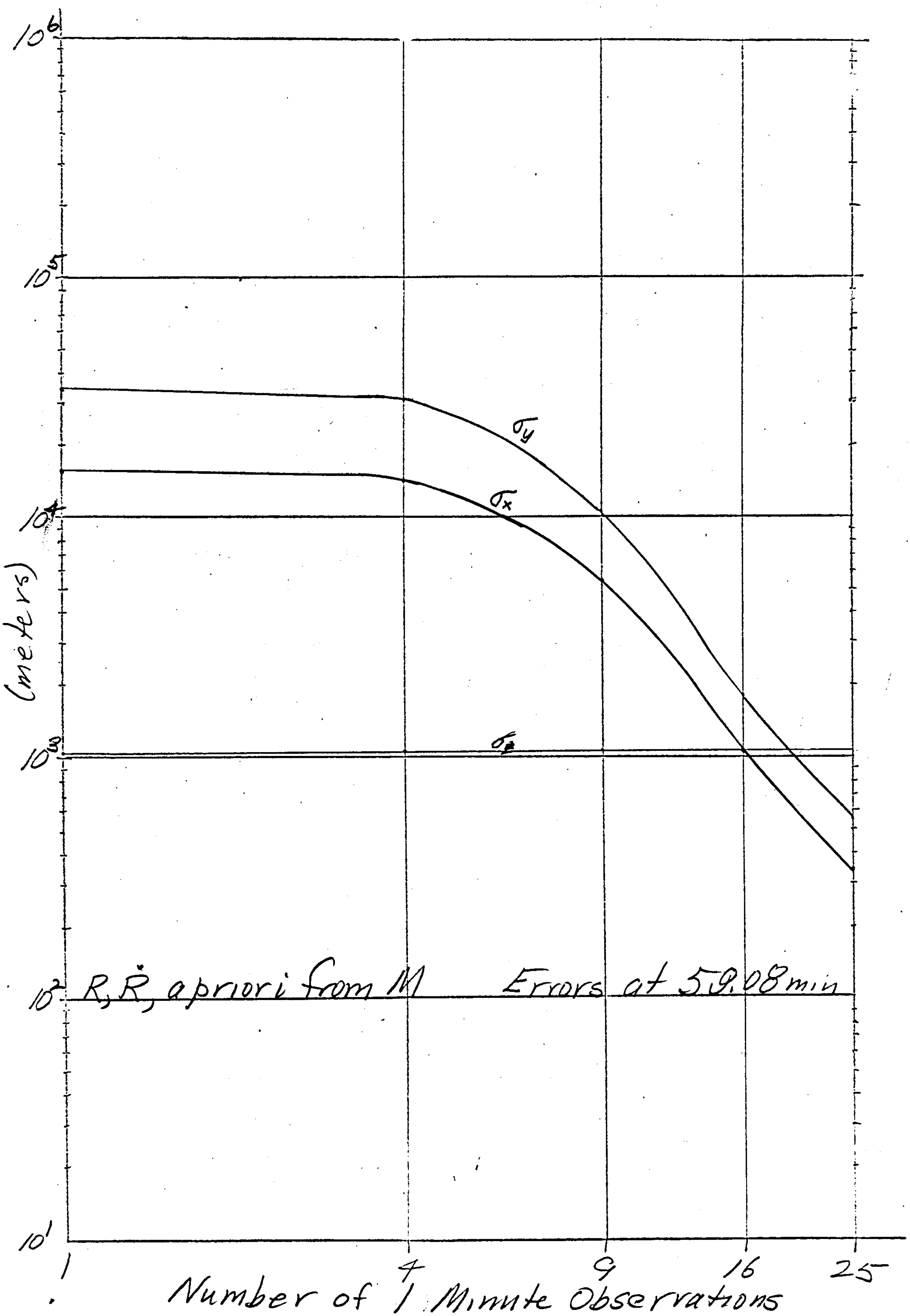
RMS Error at 59.08 min
M, J
(meters)
10^7
10^6
10^5
10^4
10^3
10^2
R
Ṙ
R, Ṙ
R
R, Ṙ
Ṙ
NO APRIORI
WITH APRIORI
1
4
9
16
25
Number of 1 Minute Observations

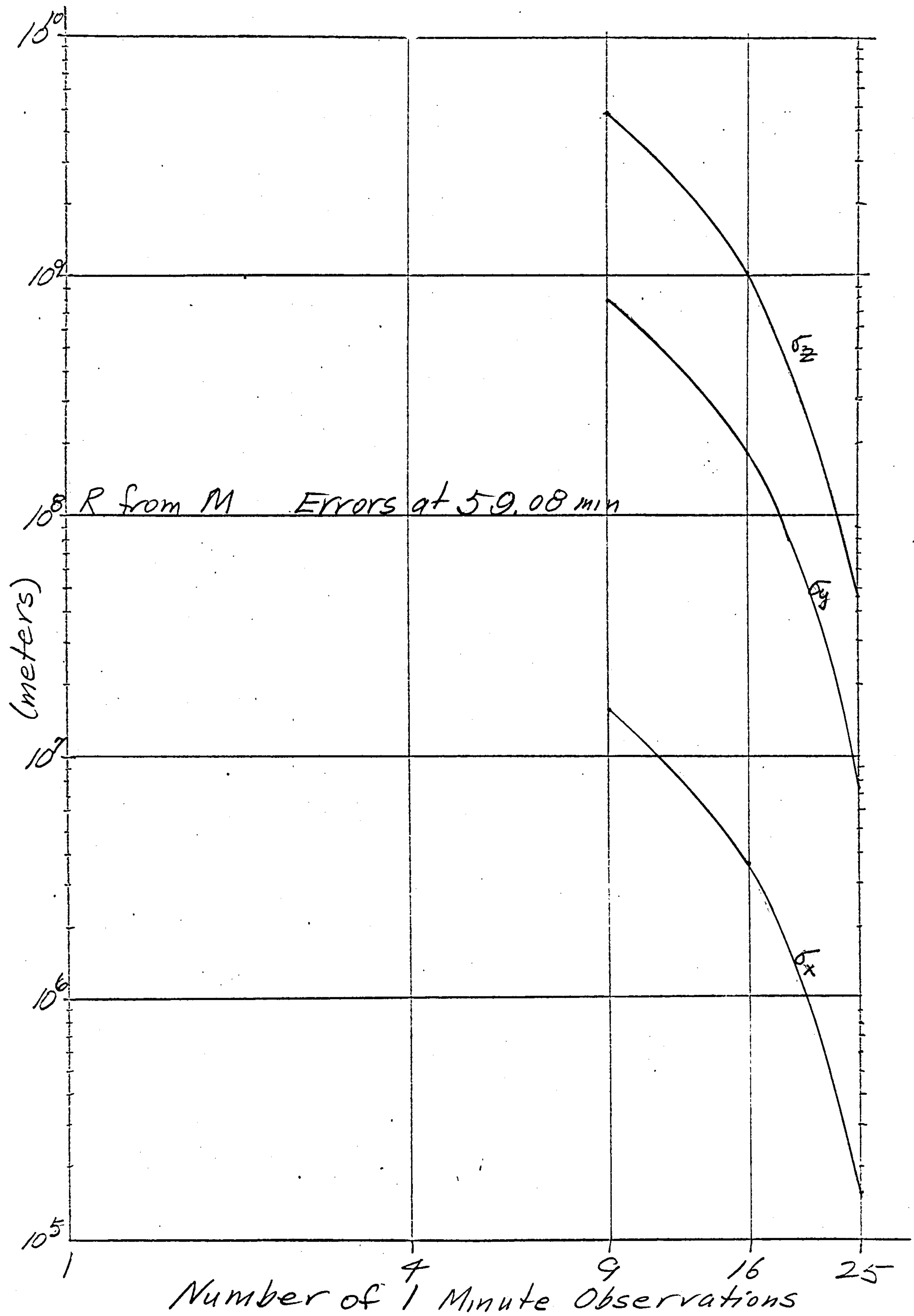

1000
100
10
1
.1
τ (seconds)
NO APRIORI
WITH APRIORI
R
Ṙ
R,Ṙ
RMS Time Between Nominal Rendezvous and Closest Approach. M,J
1
4
9
16
25
Number of 1 Minute Observations



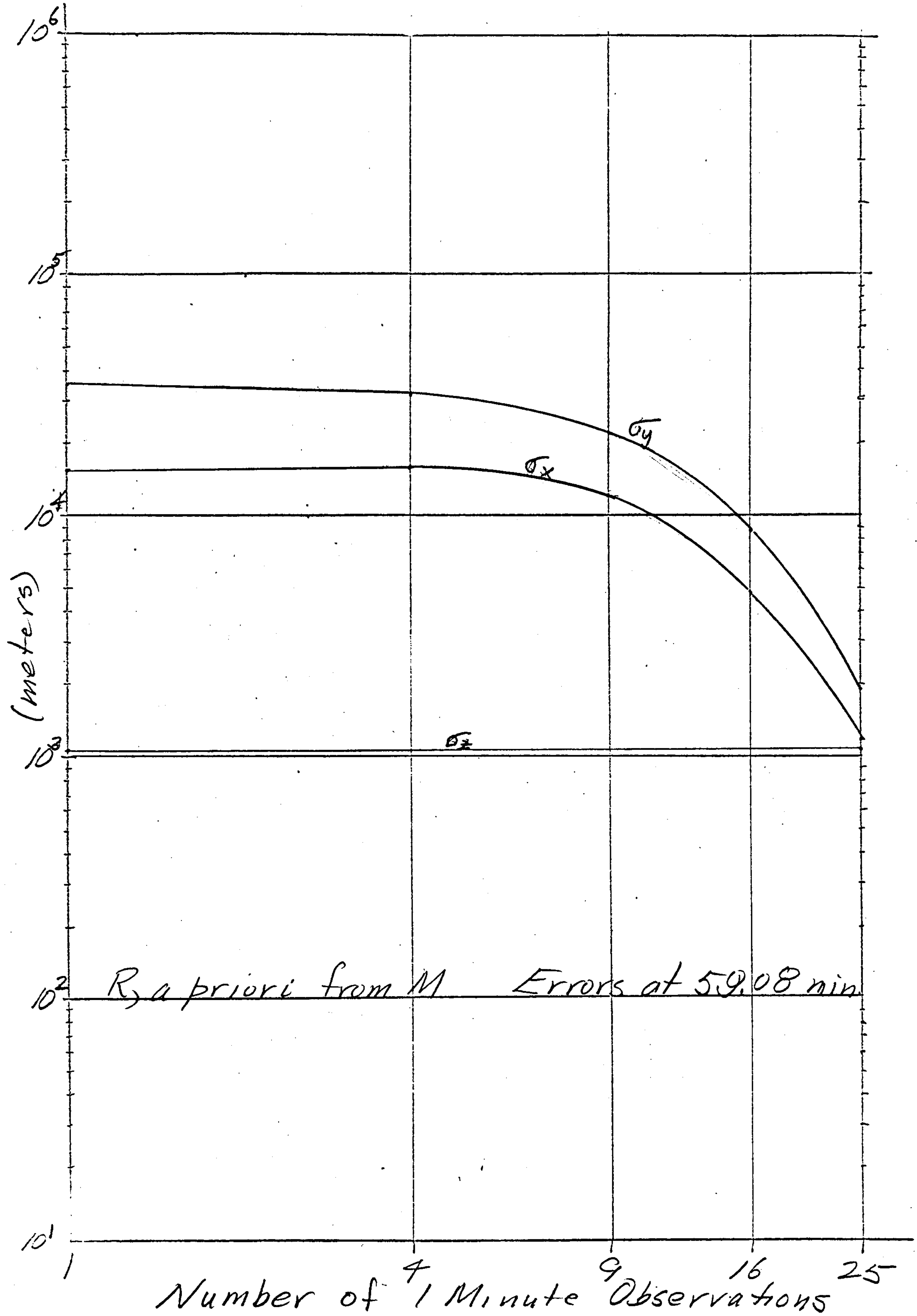
10^7
10^6
10^5
10^4
10^3
10^2
(meters)
No a priori
A priori
R
Ṙ
R, Ṙ
R, Ṙ
R
Ṙ
Minimum RMS Error
Near Rendezvous. M, J
1
4
9
16
25
Number of 1 Minute Observations


10^9
10^8
10^7
10^6
10^5
10^4
R, Ṙ from M Errors at 59.08 min.
(meters)
σ_z
σ_y
σ_x
1
4
9
16
25
Number of 1 Minute Observations

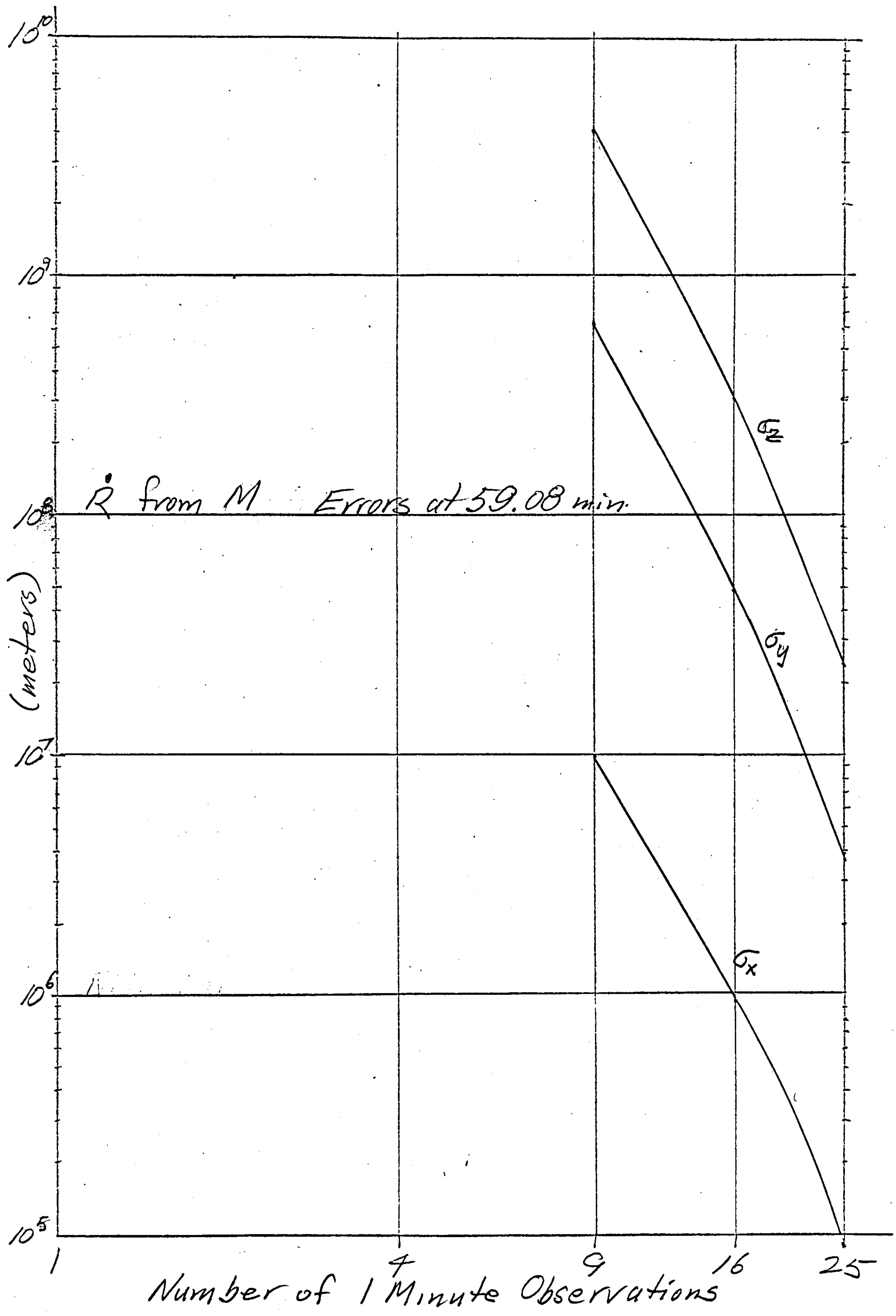

10^6
10^5
10^4
10^3
10^2
10^1
(meters)
σ_y
σ_x
σ_z
$R, \dot{R}$, apriori from M
Errors at 59.08 min
1
4
9
16
25
Number of 1 Minute Observations



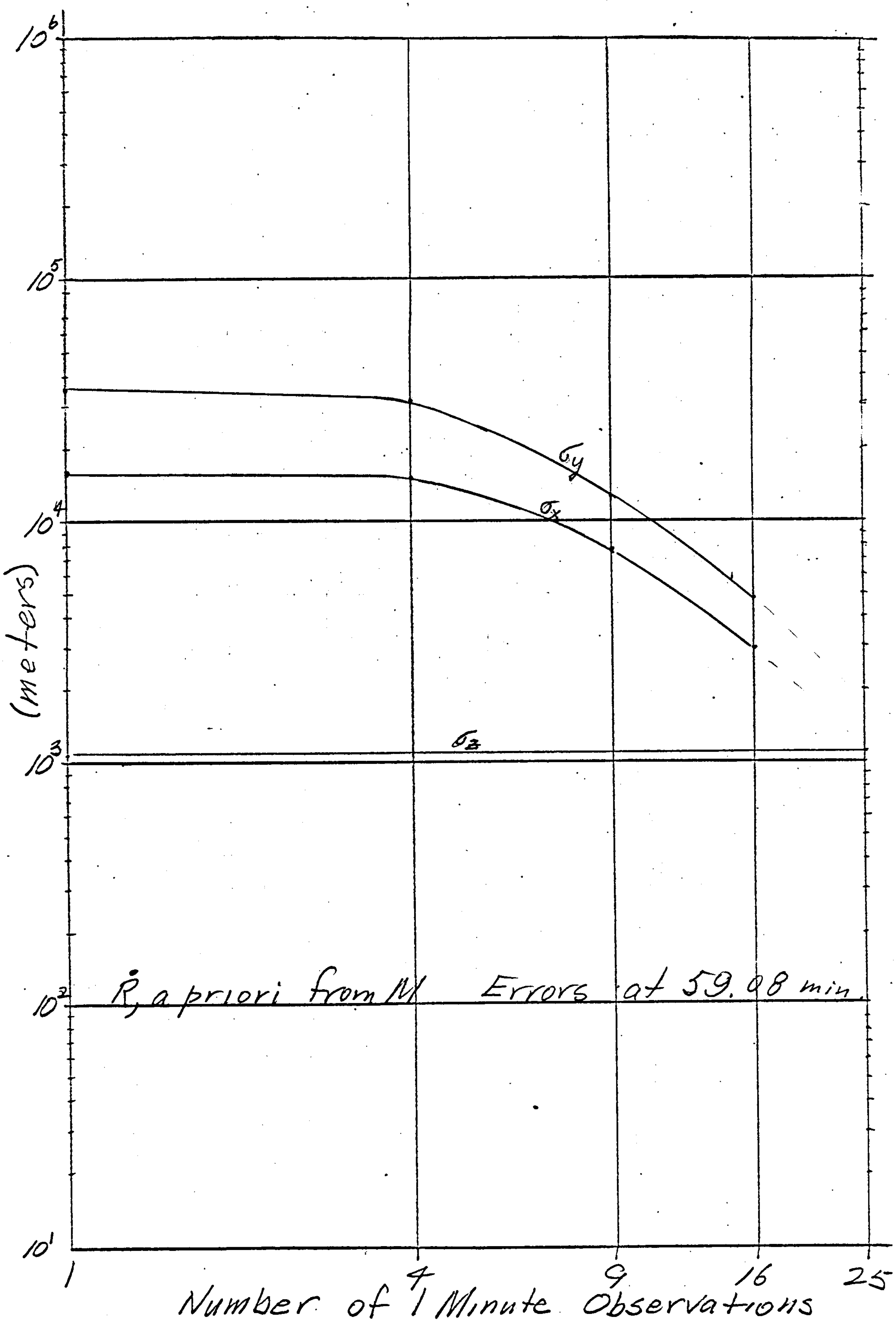
R from M Errors at 59.08 min
(meters)
10^{10}
10^{9}
10^{8}
10^{7}
10^{6}
10^{5}
σ_z
σ_y
σ_x
1
4
9
16
25
Number of 1 Minute Observations



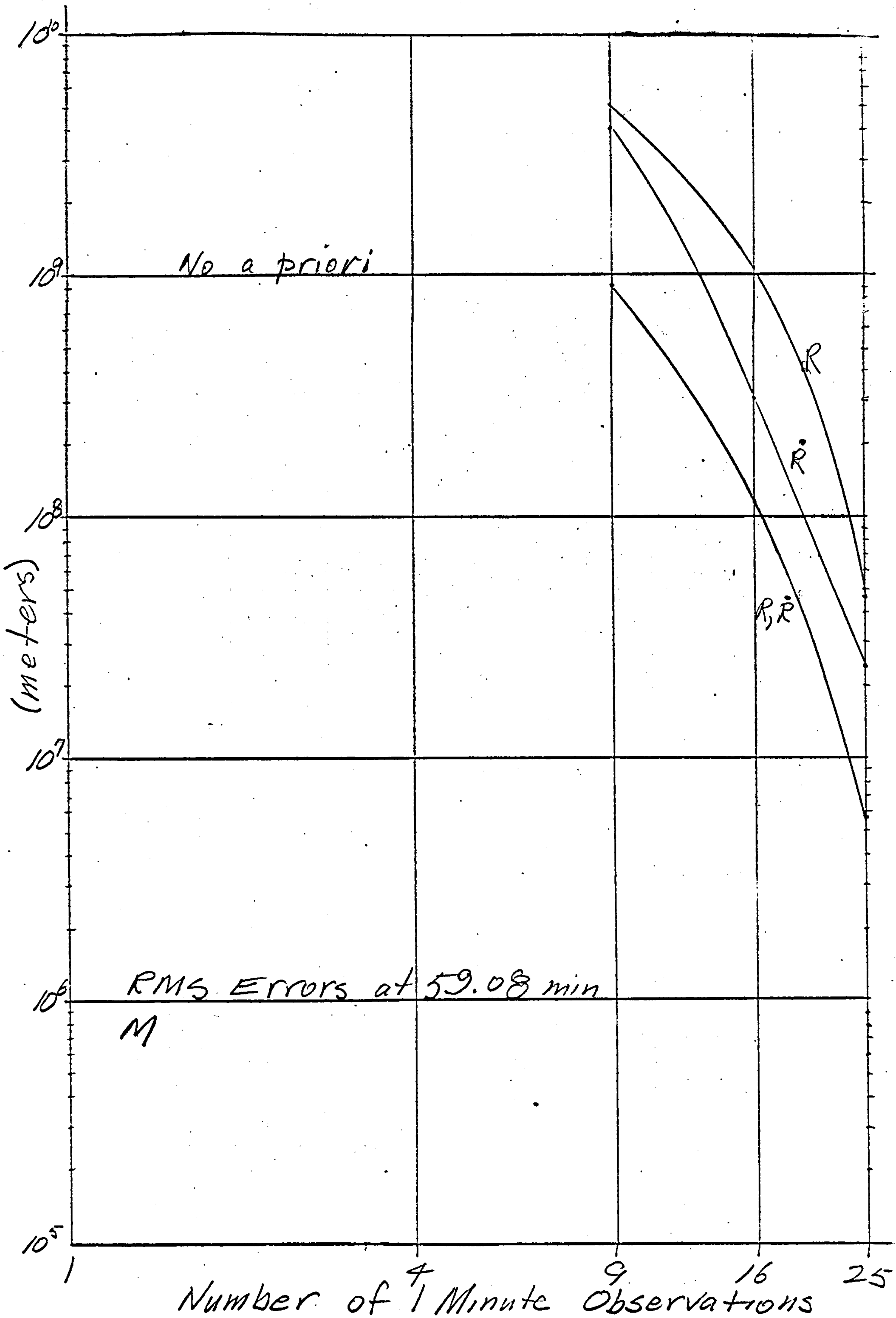
10^6
10^5
10^4
10^3
10^2
10^1
(meters)
σ_y
σ_x
σ_z
R, a priori from M
Errors at 59.08 min
1
4
9
16
25
Number of 1 Minute Observations



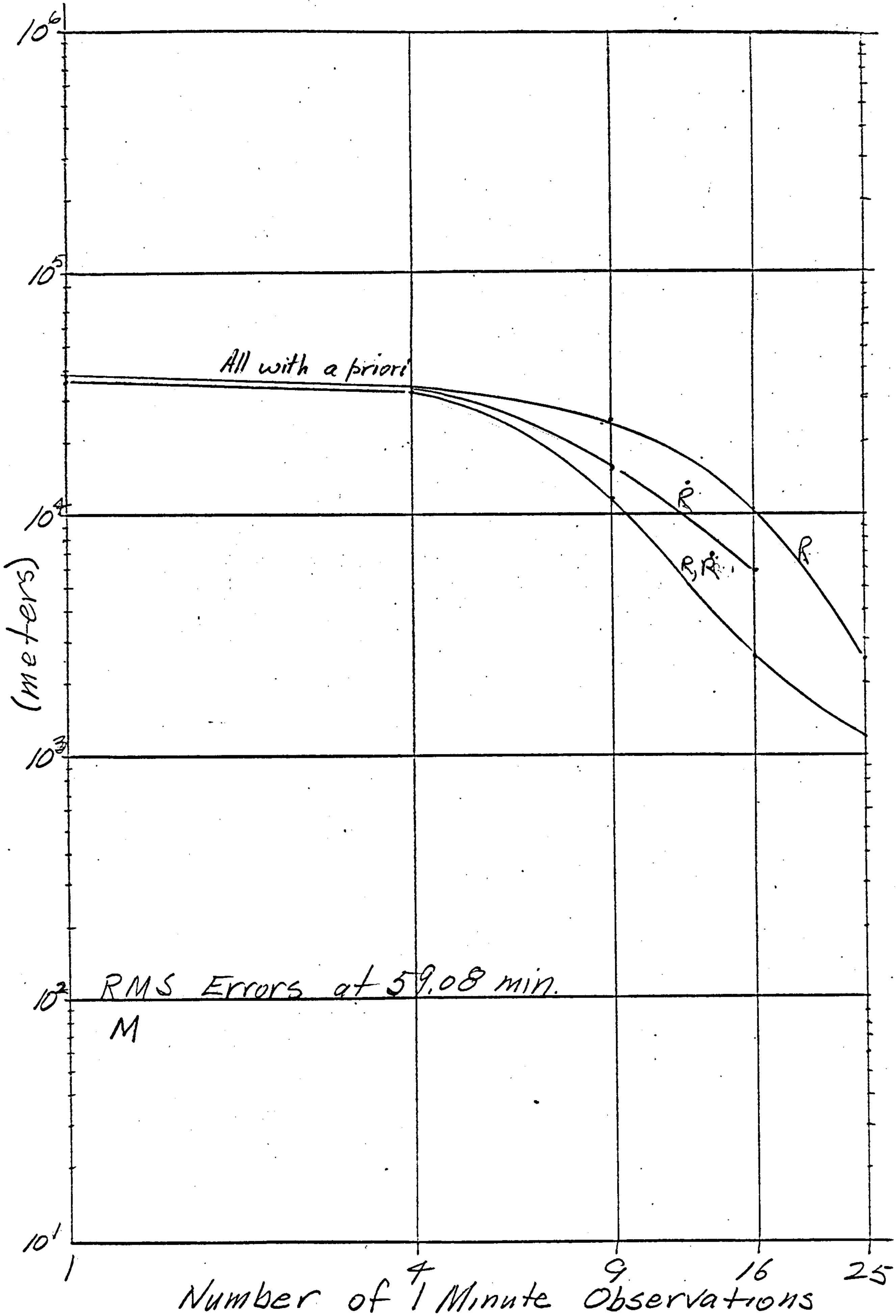
$\dot{R}$ from M   Errors at 59.08 min.
(meters)
10^{10}
10^{9}
10^{8}
10^{7}
10^{6}
10^{5}
σ_z
σ_y
σ_x
1
4
9
16
25
Number of 1 Minute Observations



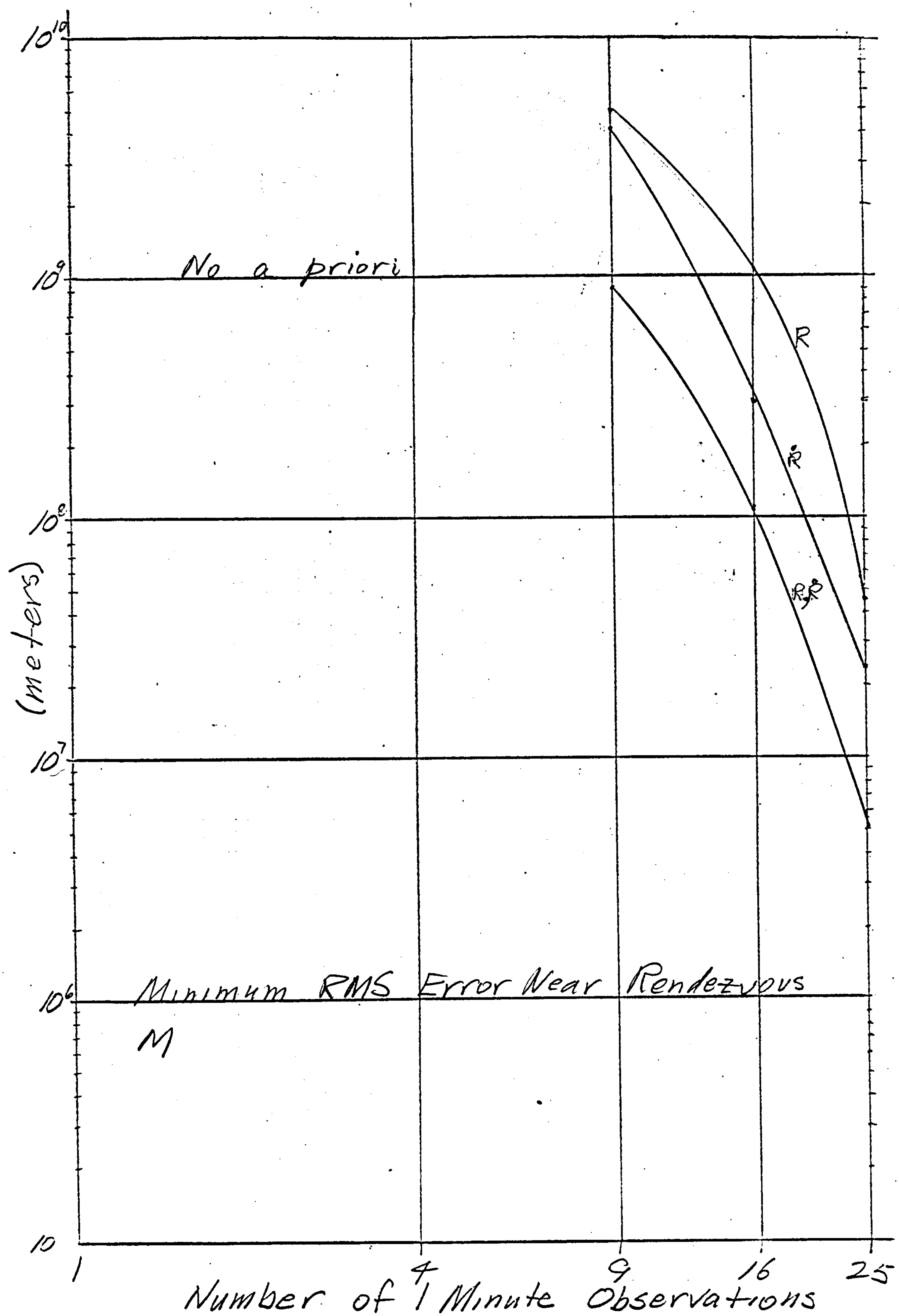
10^6
10^5
10^4
10^3
10^2
10^1
(meters)
σ_y
σ_x
σ_z
$\dot{R}$, a priori from M Errors at 59.08 min.
1
4
9
16
25
Number of 1 Minute Observations

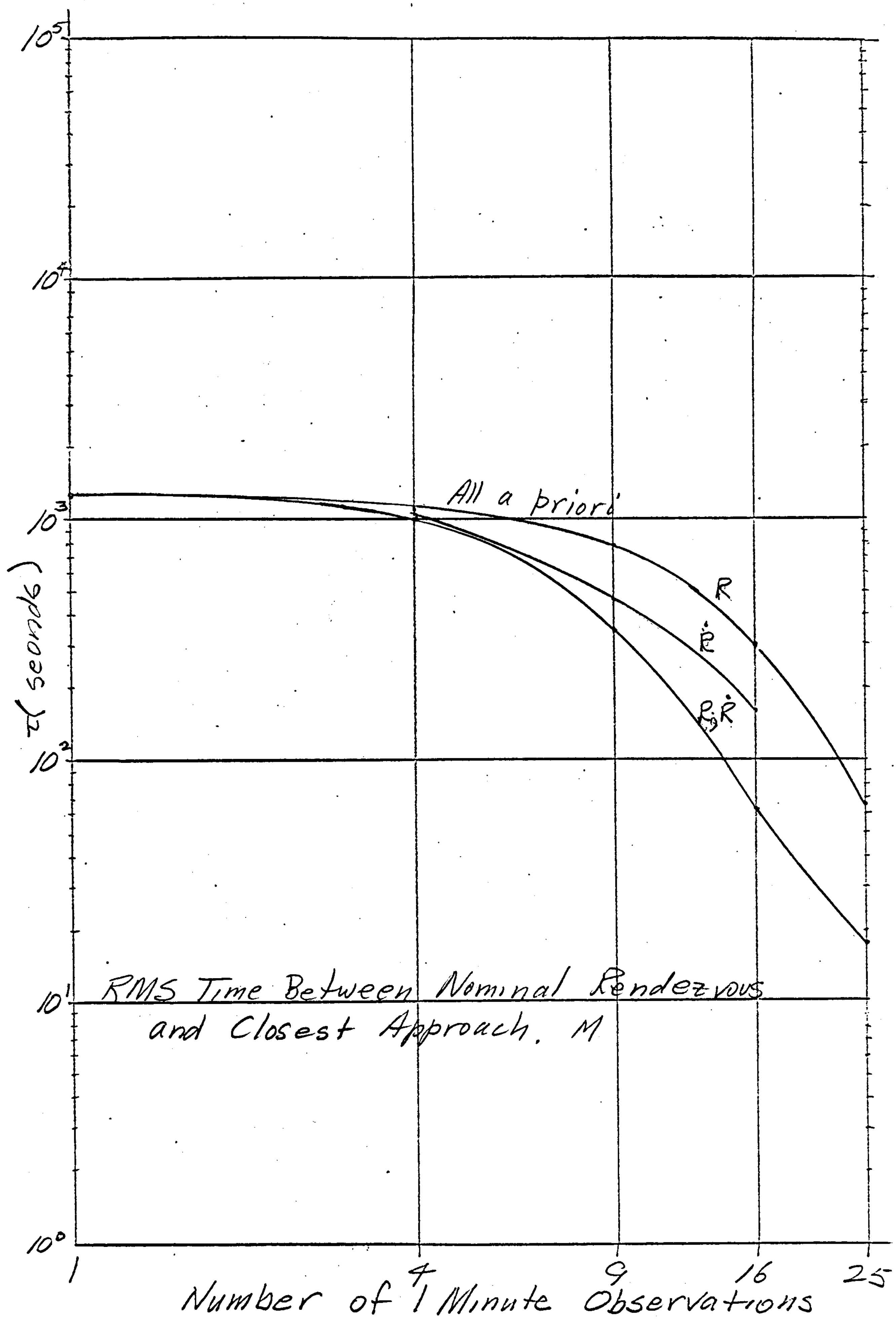


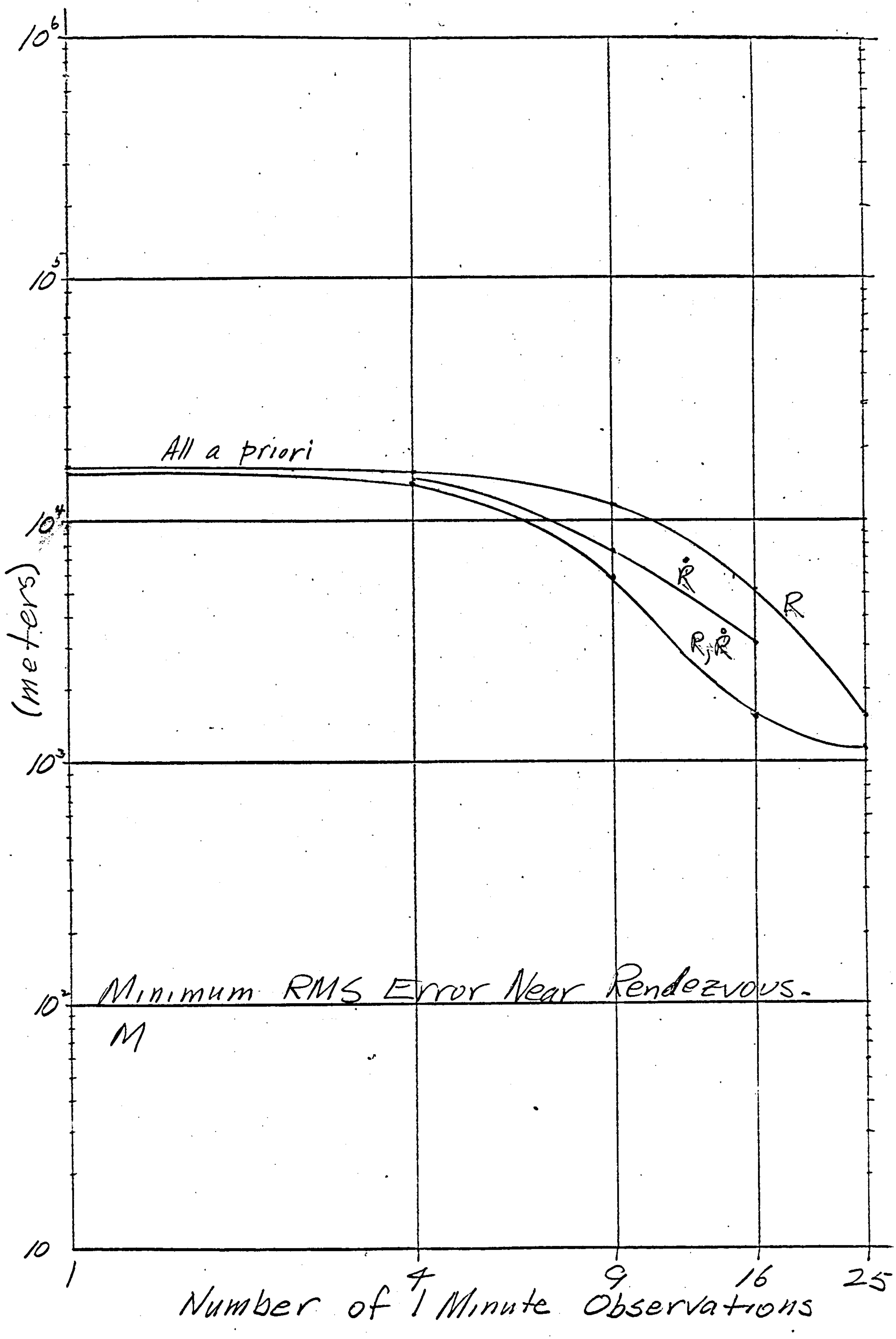
No a priori
RMS Errors at 59.08 min
M
R
Ṙ
R,Ṙ
(meters)
10^10
10^9
10^8
10^7
10^6
10^5
1
4
9
16
25
Number of 1 Minute Observations



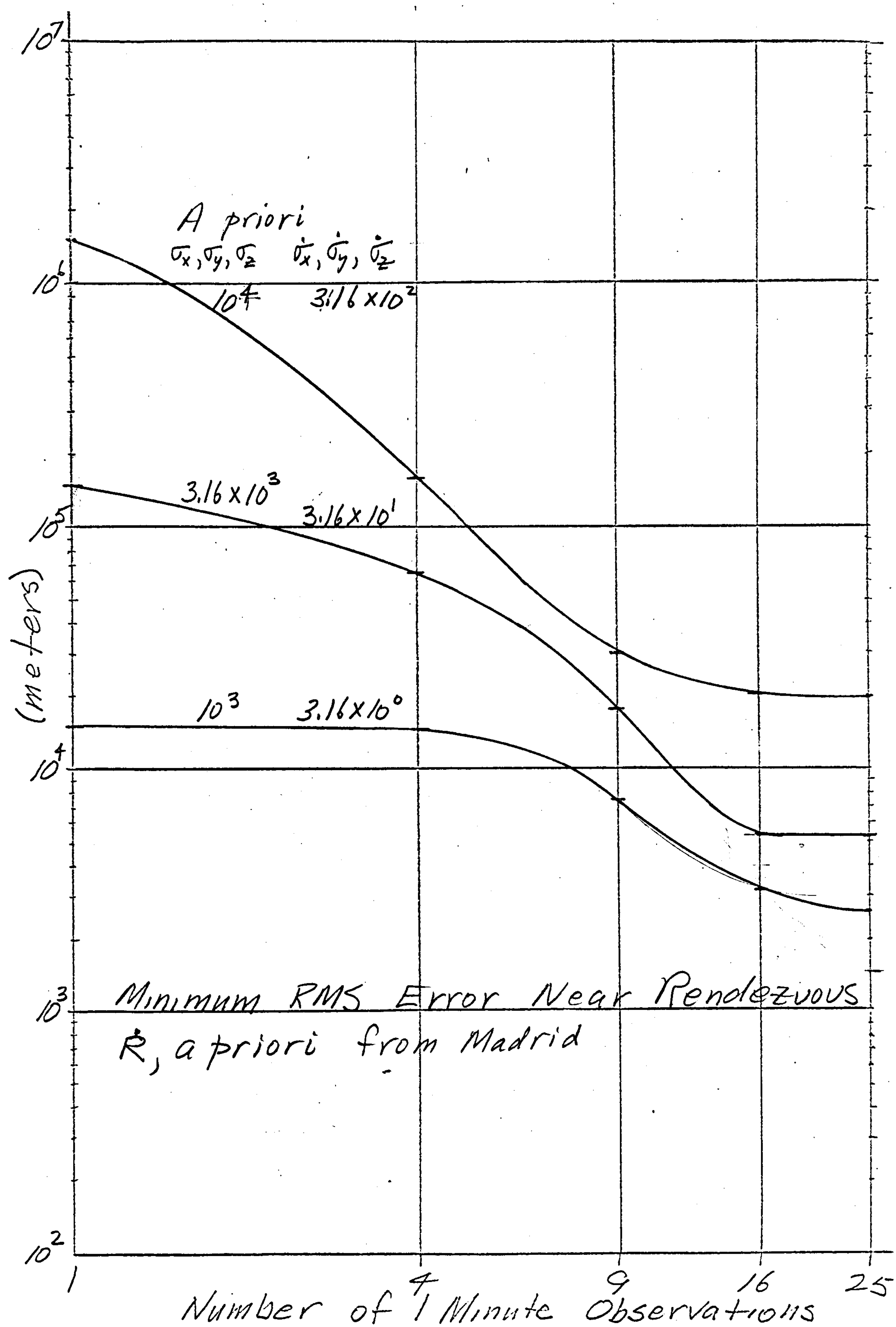
10^6
10^5
10^4
10^3
10^2
10^1
(meters)
All with a priori
$\dot{R}$
R
$R, \dot{R}$
RMS Errors at 59.08 min.
M
1
4
9
16
25
Number of 1 Minute Observations


No a priori
Minimum RMS Error Near Rendezvous
M
(meters)
R
Ṙ
R, Ṙ
Number of 1 Minute Observations


10^5
10^4
10^3
10^2
10^1
10^0
τ (seconds)
All a priori
R
$\dot{R}$
R, $\dot{R}$
RMS Time Between Nominal Rendezvous and Closest Approach, M
1
4
9
16
25
Number of 1 Minute Observations



10^6
10^5
10^4
10^3
10^2
10
All a priori
$\dot{R}$
R
$R, \dot{R}$
(meters)
Minimum RMS Error Near Rendezvous - M
1
4
9
16
25
Number of 1 Minute Observations



Minimum RMS Error Near Rendezvous
$\dot{R}$, a priori from Madrid

PART II

FAR-SIDE RELAY

PART II: FAR-SIDE RELAY

INTRODUCTION TO PART II

Communication with the CM/SM or LEM from Earth is interrupted while these vehicles are behind the Moon. Some of the critical mission phases, however, occur on the far-side of the Moon. These include injection of the CM/SM into lunar orbit, rendezvous between CM/SM and LEM, and injection of CM/SM into return-to-Earth orbit. It is desirable to be able to communicate with the Apollo vehicles at these critical times.

This can be accomplished by using the S-IVB as a radio relay vehicle by properly boosting it after separation from the CM/SM so that it is within communication distance of the Earth and the vehicles behind the Moon during the time from before injection of the CM/SM into a lunar orbit until after the CM/SM is injected into a return-to-Earth trajectory.

Narrow-band communication or radar requirements are discussed in Apollo Note No. 35, Lunar Far-Side Relay Technique - Basic Radar Considerations* and in another report.[7/] Apollo Note No. 44, Back of Moon Relay Trajectories* starts the search for suitable trajectories, and further treats narrow-band communication requirements. Section 7 of Apollo Note No. 87, Rendezvous Aids - Far-Side Relay*, Apollo Note No. 90, Further Examination of Far-Side Relay Trajectories*, and Apollo Note No. 97, Minimum Boost Velocity Requirements for Far-Side Relay*, examine further the boost capabilities required of the S-IVB for this purpose.

7/ Bissett-Berman Corporation Report C60-6, DSIF Capability for Apollo Guidance and Navigation.

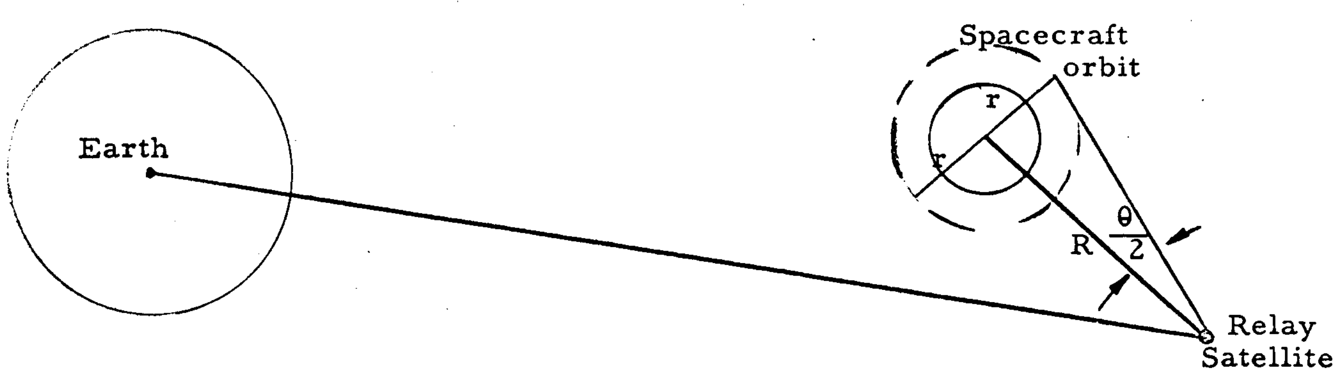
APOLLO NOTE NO. 35

H. Epstein
21 March 1963

LUNAR FAR SIDE RELAY TECHNIQUE - BASIC RADAR CONSIDERATIONS

This note deals with some fundamental radar considerations involved in a relay technique suggested by Mr. L. Lustick. The purpose of the relay is to make possible communications between a spacecraft on the far side of the moon and the DSIF. The relay satellite is injected into a solar orbit from the Apollo spacecraft prior to a lunar parking orbit being established on the translunar trajectory. This expendable satellite is needed only to provide a relay function until the CM is placed on its transearth trajectory. Considerations involved in the selection of a suitable relay trajectory will be the subject of a future note (i. e., geometry and fuel considerations).

One feasible approach to allow trajectory measurements to be made while the Apollo spacecraft is occulted from the earth by the moon is indicated in the material that follows. The geometric aspects are illustrated in the figure below.



During the time that the spacecraft is occulted from the earth by the moon, the transponder directional antenna system is pointed towards the relay satellite. During this time, the DSIF antennas are then made to look in the direction of the relay satellite. The relay satellite is postulated as possessing two antenna systems; one antenna system being employed between the DSIF and the relay, and the other system for communication between the spacecraft and the relay satellite. The most severe conditions at large distances will be associated with the link between the relay satellite and the spacecraft. The transponder characteristics for both the spacecraft and relay satellite are taken from Apollo Note No. 18. Two antennas are postulated for the relay to minimize power requirements and yet be compatible with simple antenna pointing system as already in use. One antenna would be essentially omnidirectional for the near lunar phase and the other as a four-foot parabolic dish for the deeper phase operation. The four-foot dish has an antenna beamwidth of about 7.5° and would be used at distances in excess of about 40,000 KM. A rather simple optical seeker could be employed with an accuracy (3 σ) of about 2 degrees. The radar angular coverage to encompass the angular region involved in the spacecraft orbit is given by:

$$\theta \cong 2 \tan^{-1}\left(\frac{r}{R}\right)$$

This function is plotted in Figure 1. For distances in excess of 40,000 KM a 2 degree pointing accuracy would reduce the antenna gain by no more than about 3 db.

The received power and signal-to-noise ratio will now be considered. The received power (P_R) is given by:

$$P_R = P_T G_A G_R \left(\frac{\lambda}{4 \pi R}\right)^2 L_Z$$

P_T = transmitter power output = .2 watt (solid state transmitter)

G_A = gain of Apollo antenna = 26.5 db (4 foot dish)

G_R = gain of relay antenna = 26.5 db (4 foot dish)
= 0 db (omni - antenna)

λ = free-space wavelength = 13 cm

L_Z = total losses = 0 db (in practice this number will lie between 3 - 10 db for typical system)

R = distance between Apollo spacecraft and relay satellite (assumed equal to distance from relay satellite to center of moon)

Numerical values are indicated in Figure 2. The level of -150 dbm corresponds to the receiver noise level associated with a 11 db noise figure receiver and a bandwidth of 20 cps and is about the threshold level of a phase lock loop. The omni-antenna system would yield a 10 db signal-to-noise ratio up to distances of about 40,000 KM. For longer distances, the 4 foot parabolic dish would be employed. Useful ranges of about 10^6 KM would be obtained. The 40,000 KM distance was earlier indicated as compatible with this antenna system and the two degrees of pointing accuracy. Broader bandwidth transmissions and system losses can be accommodated by combination of increased power levels, improved noise figures, and larger antenna systems. Much larger antenna systems would impose a necessity for automatic RF tracking techniques to be employed. The minimum bandwidth that can be employed is determined by the rate of change of doppler frequency. A rate of change of range rate corresponding to about 2 moon g's at S-band would place a minimum bandwidth restraint of about 5 cps which is fully compatible with the minimum 20 cps bandwidth considered earlier.

One additional radar consideration should be mentioned; namely, CW waveforms are employed in this mode. Neglecting time sharing possibilities, it will be necessary that two transponders be employed in the relay satellite since four distinct frequencies are involved simultaneously (two for transmission and two for reception). The

Apollo spacecraft transponder may remain unchanged provided that the DSIF utilizes two sets of transmitting and receiving frequencies (not required simultaneously). This would necessitate additional frequency allocations for the DSIF. If the DSIF was maintained unchanged, two sets of transponders would be required by the Apollo transponder (some common circuitry could be employed).

In summary, from radar consideration alone, the relay technique is quite feasible particularly where narrow bandwidths are involved.

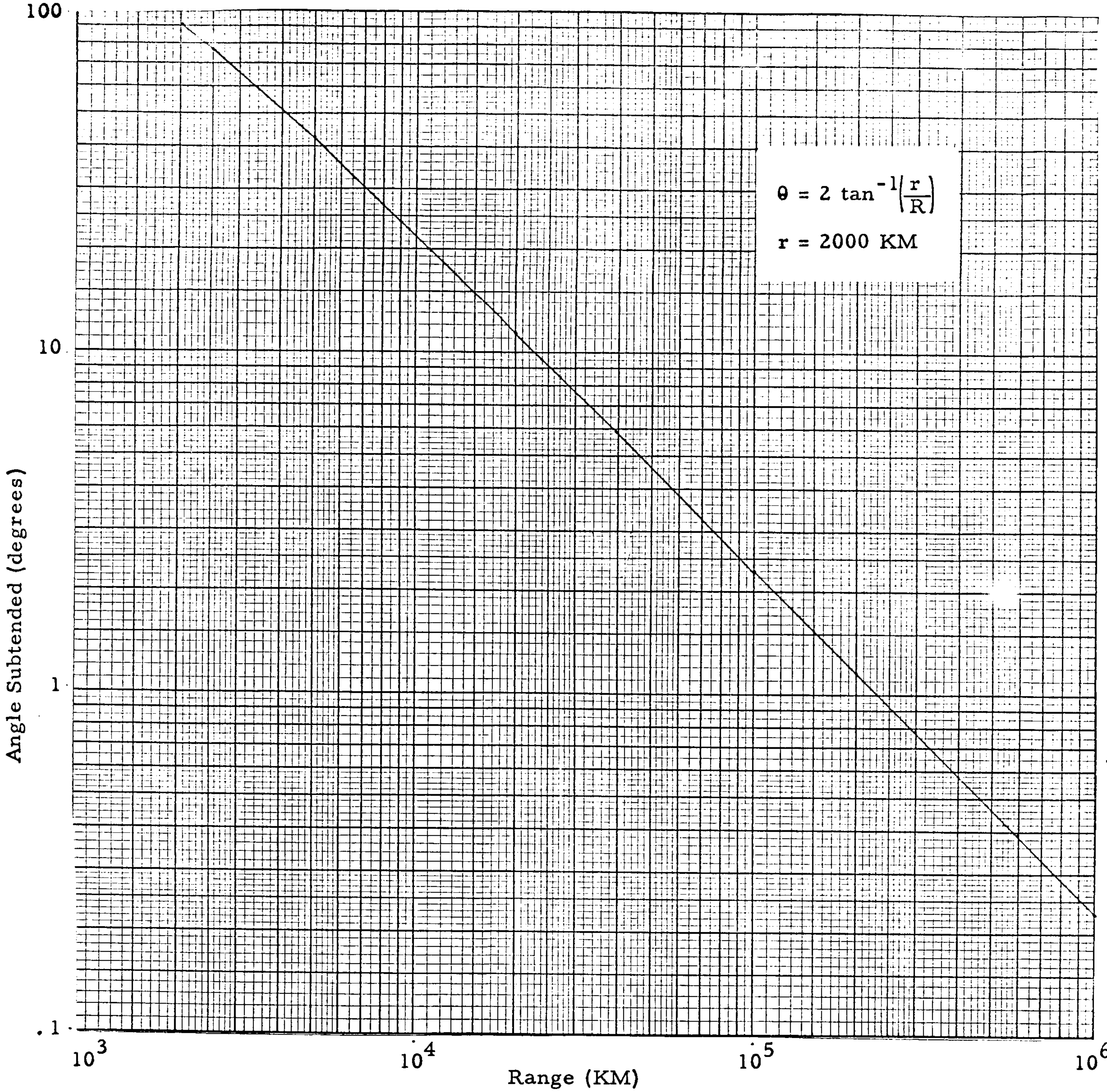


Figure 1. Radar Angular Coverage Requirement vs Relay Satellite Distance from Center of Moon

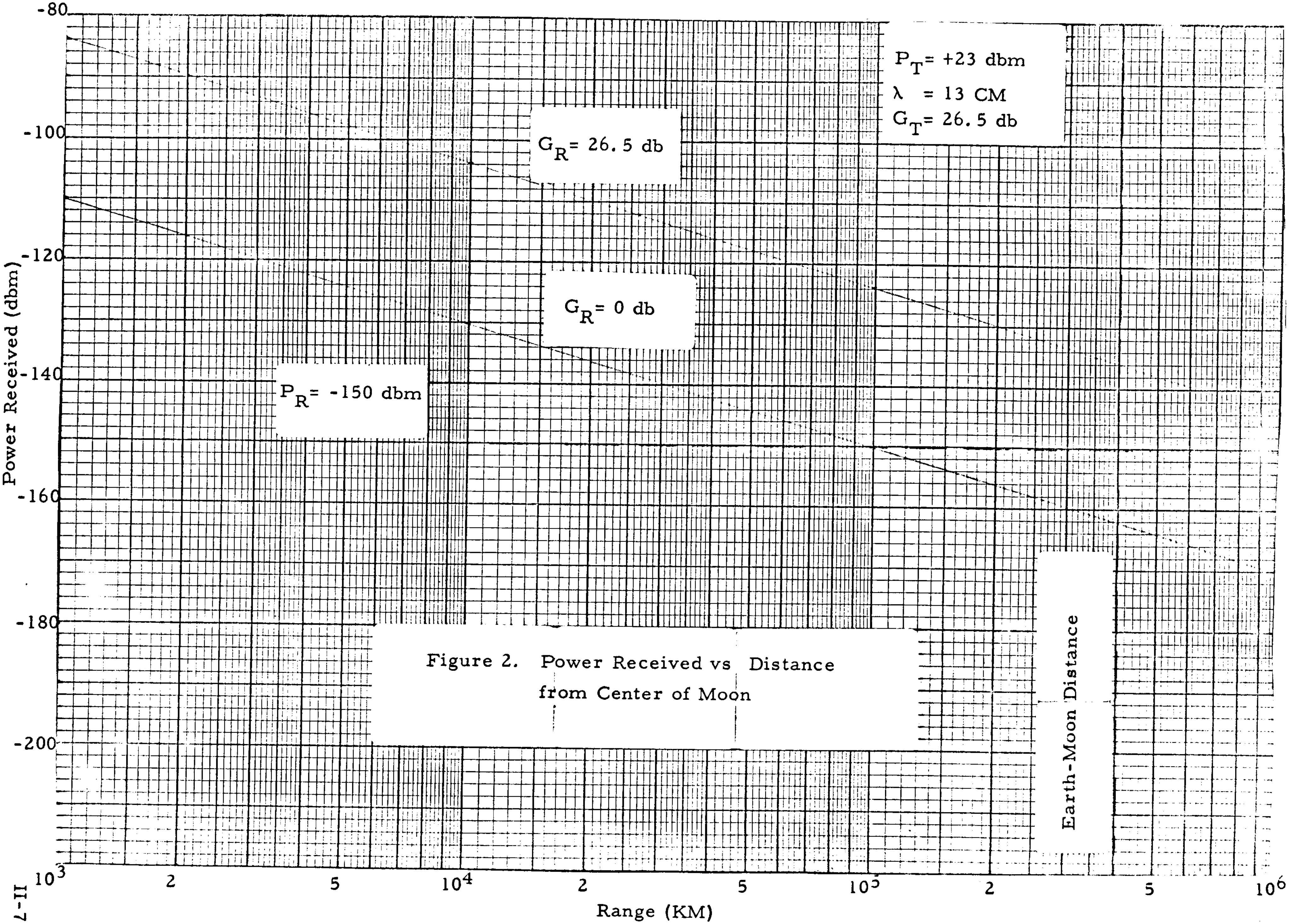


Figure 2. Power Received vs Distance from Center of Moon

APOLLO NOTE NO. 44

L. Lustick
16 April 1963

BACK OF MOON RELAY TRAJECTORIES

Purpose

The purpose of this note is to examine the feasibility of establishing a relay which could be used to communicate with the Apollo vehicle while it is on the back-side of the moon.

Introduction

It has been suggested that it would be desirable to have a relay that would allow communication with a vehicle while it was on the back-side of the moon. A possible scheme for accomplishing this (suggested by Dr. Yarymovych) is to use the translunar injection vehicle (S-4-B). The(S-4-B) subsequent to translunar injection is approximately in a free return orbit to the earth. The idea is to use the pad in this vehicle to boost itself or a special purpose relay package so as to allow it to escape from the back-side of the moon. In order to evaluate if the idea is practical, it is necessary to establish the relay trajectory relative to the moon as a function of boost velocity. In particular, the range and the portion of the moon visible from the relay as a function of time are of interest.

Method

1. The time and angular travel for a vehicle with respect to the earth to reach a distance from the earth equal to the mean distance from earth to moon were calculated as a function of injection velocity. The effect of the moon was neglected and the injection angle and altitude were held constant at 20 degrees and 200 nautical miles, respectively.

2. The injection velocity consistent with a trip time of approximately 72 hours was selected as the standard orbit.

3. The point where the standard orbit would be required to pierce the lunar sphere of influence in order to have a perilune altitude of approximately 100 nautical miles was next determined.

4. The effect of increments of boost velocity on impact conditions with the LSOI was established.

5. The trajectory of the relay with respect to the moon was established for boost velocities of 160, 260, 360 ft/sec.

Results

Figure 1 is a plot of the trip time as a function of the translunar injection velocity. It can be seen from Figure (1) that the trip time is a very strong function of the translunar injection velocity. For an injection velocity of 35,440 ft/sec., the trip time is 69.5 hours and this condition was selected as the standard orbit.

The impact location and hyperbolic orbit relative to the moon for the standard orbit are shown in Figure 2. It should be noticed that in order to have a perilune of approximately 100 nautical miles it is necessary to impact the lunar sphere at a point 51.5 degrees removed from the line connecting the earth and the moon.

Figure 3 is a plot of the angular travel of the vehicle with respect to the earth as a function of the translunar injection velocity. The angular travel is consistent with a radial distance from the earth equal to the mean distance from earth to the moon.

Figure 4 is a plot of the change in the angular location of the vehicle relative to the line connecting the earth and the moon from the angular displacement of the standard orbit as a function of boost velocities.

$$\left[(\psi - \omega_m T) - (\psi_s - \omega_m T_s)\right] .$$

For the boost velocities considered (160 - 360 ft/sec), the relay orbits would completely miss the lunar sphere of influence (58×10^6 meters).

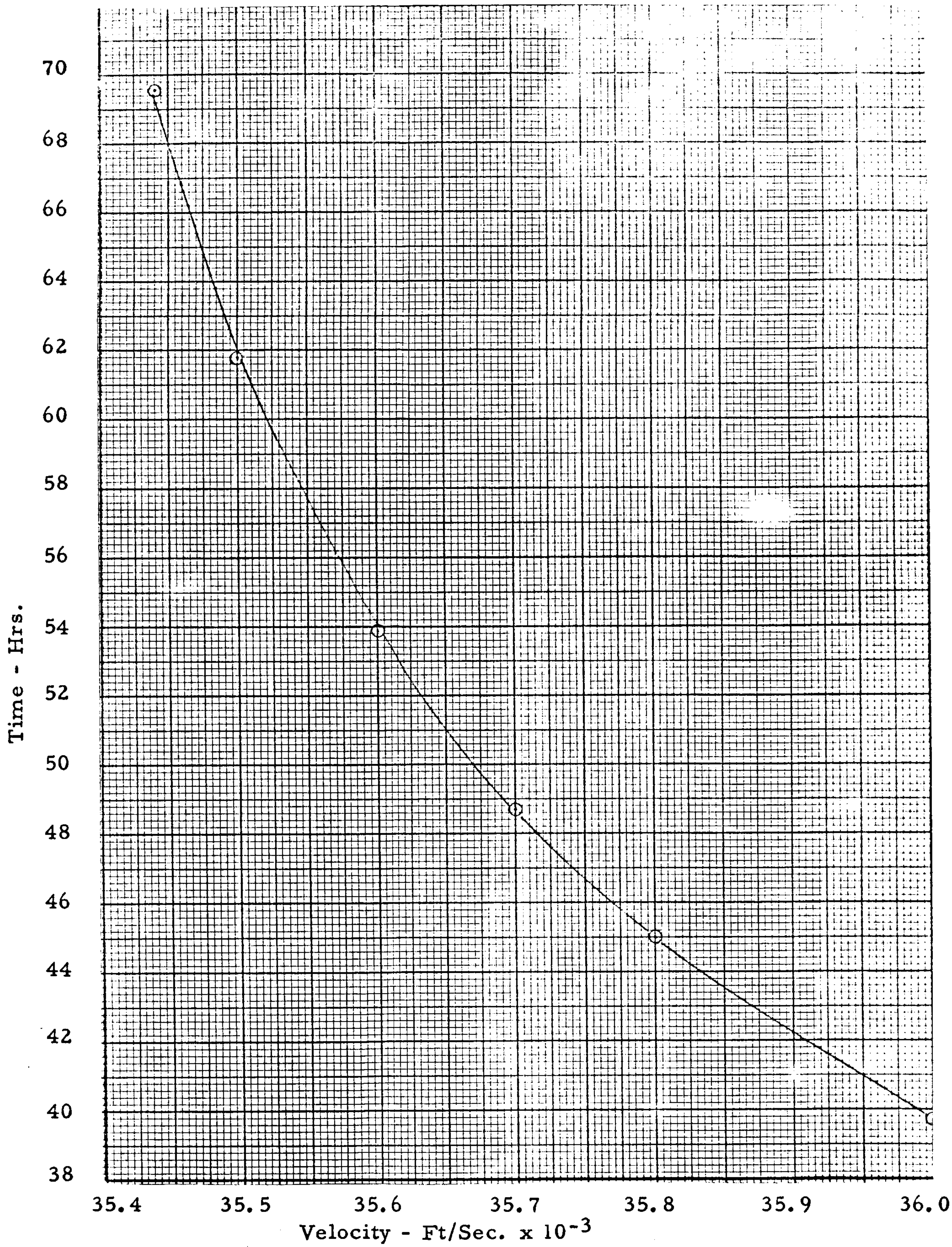


Figure 1.

Time Versus Injection Velocity
(Final Radius = Earth Moon Distance)

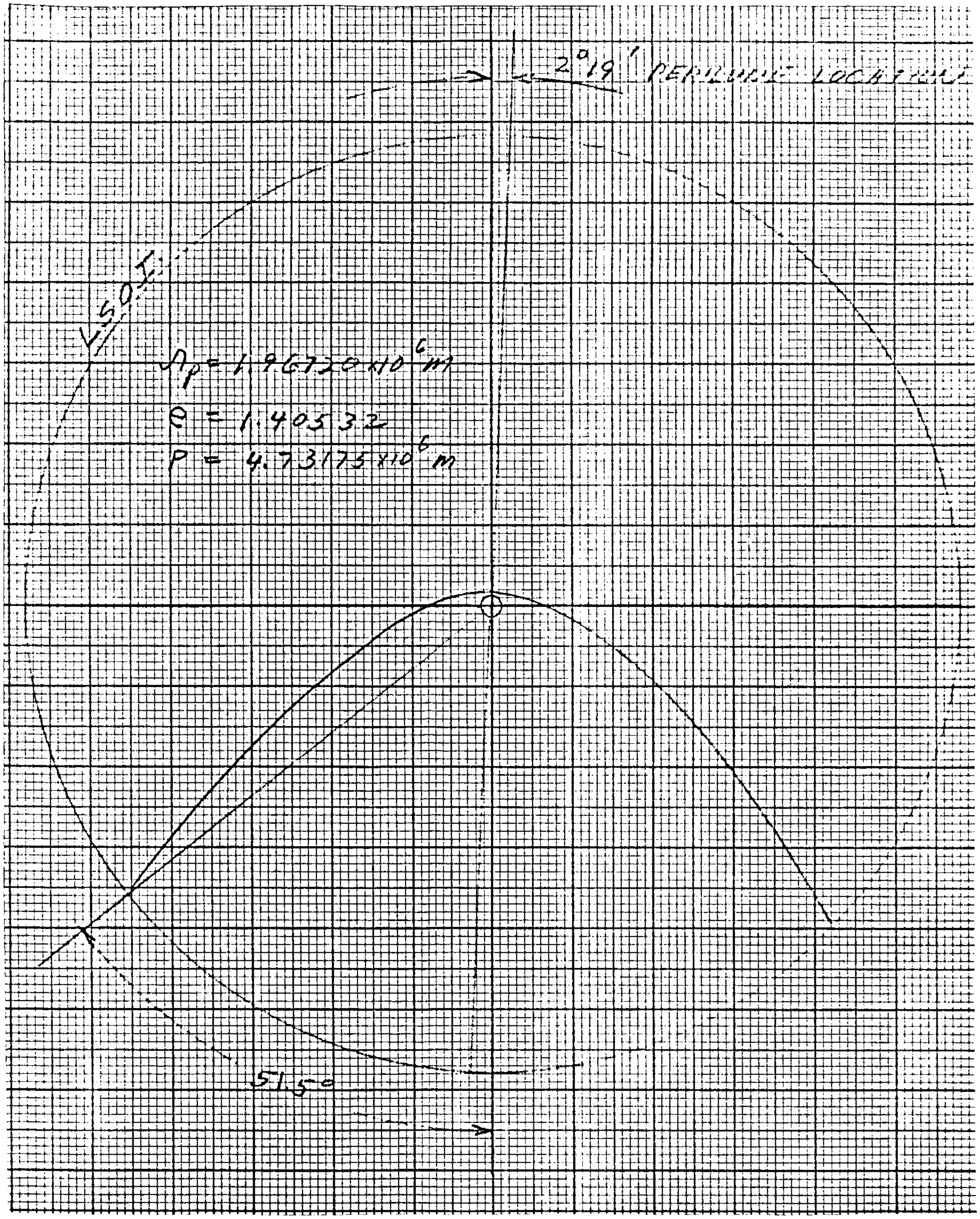


Figure 2.

Standard Hyperbolic Orbit Relative To The Moon

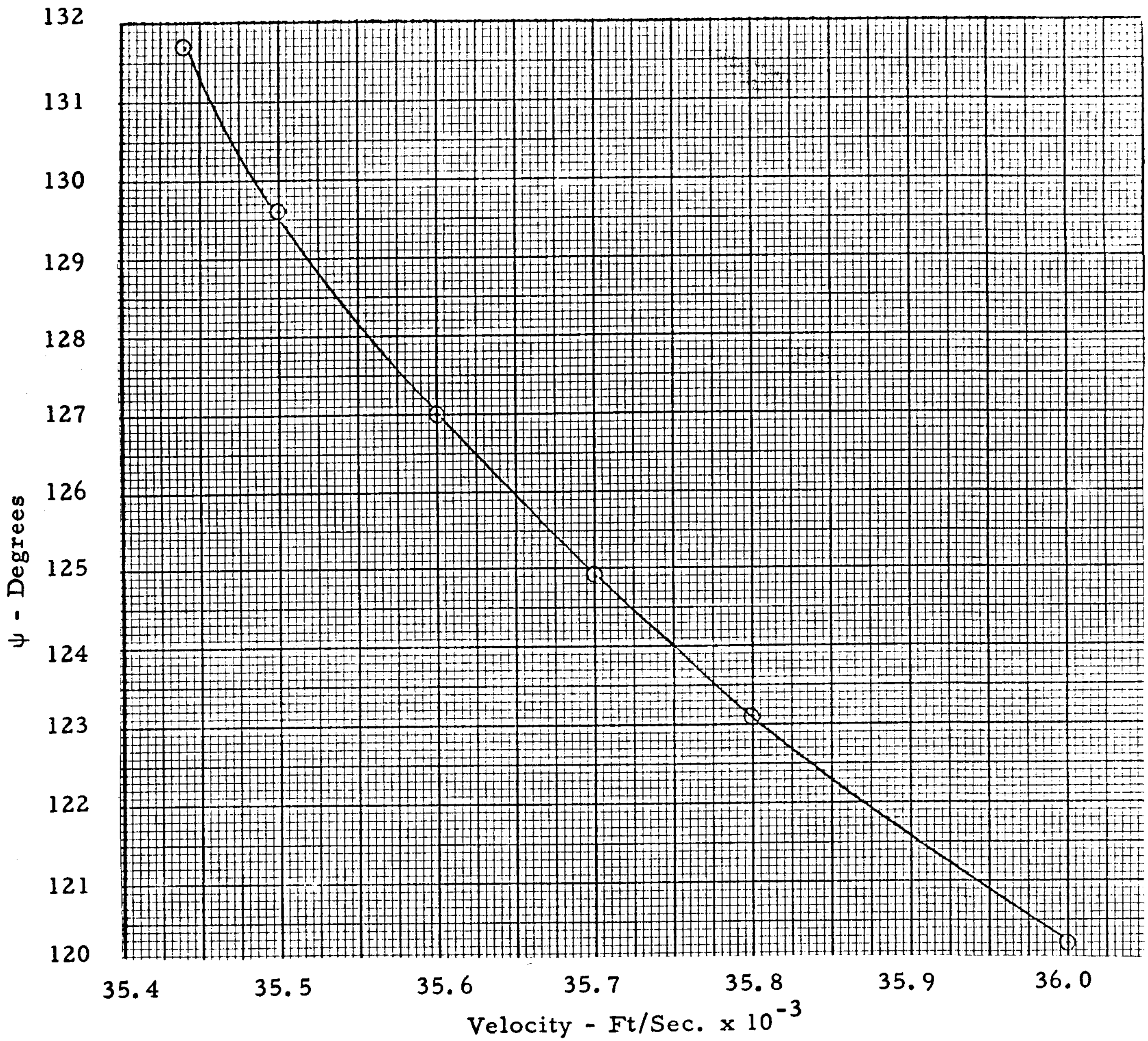


Figure 3.
Vehicle Angular Travel versus Injection Velocity
(Final Radius = Earth Moon Distance)

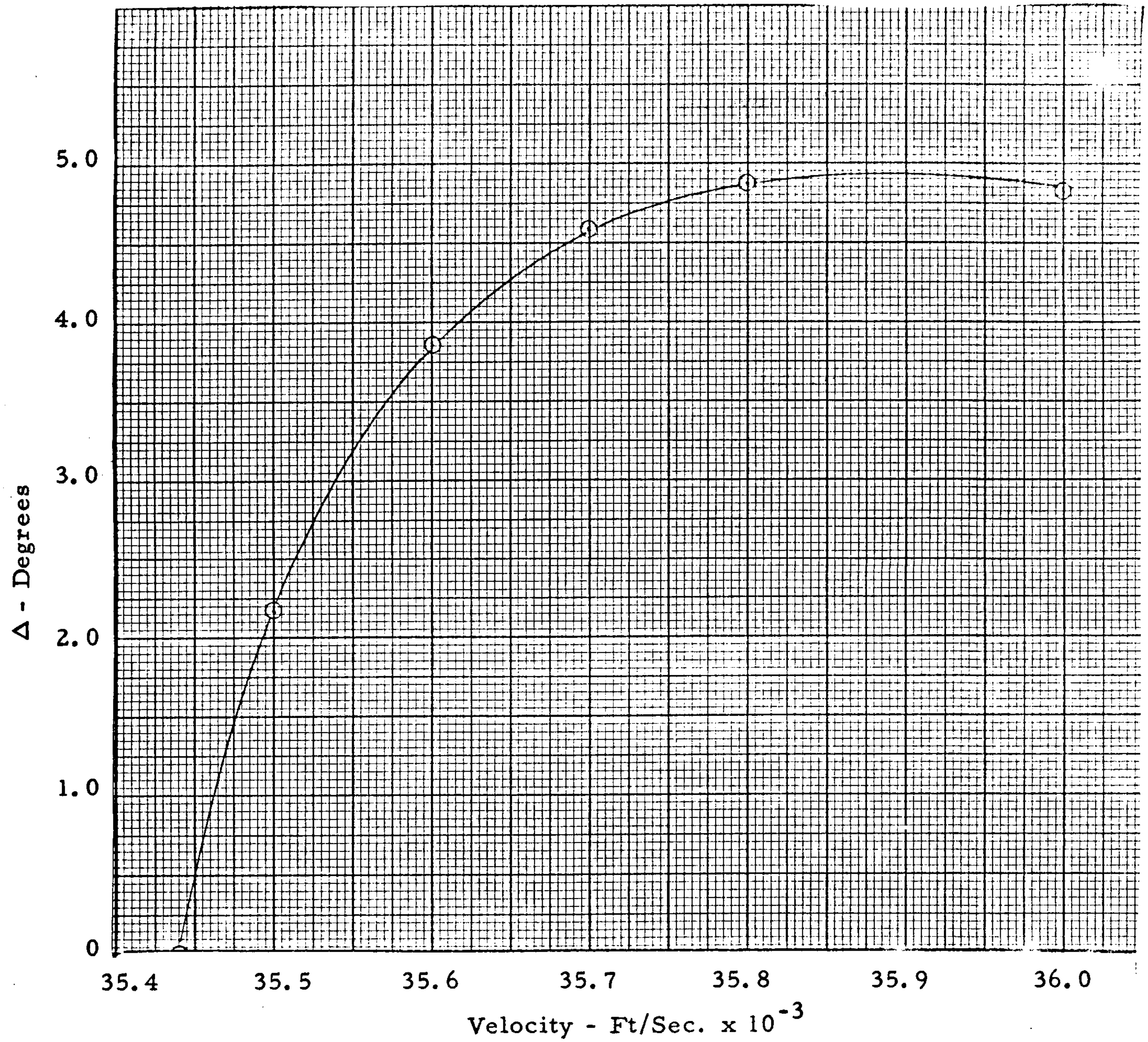


Figure 4.
Angular Displacement of the Relay Relative to the
Standard Orbit

Figure 5 is a plot of the position of the relay relative to the moon as a function of boost velocity at times of 72 and 100 hours. The range of the relay relative to the moon is less than the mean distance from earth to the moon for any of the boost velocities considered for the longest time considered (100 hours). At a time of 72 hours , approximately 153 degrees of the back-side of the moon is visible independent of boost velocity. At a time equal to 100 hours, approximately 129 and 134 degrees of the back-side of the moon are visible for boost velocities of 160 and 360 ft/sec., respectively. Of the boost velocities considered, 160 ft/sec. is most desirable since the range relative to the moon is the least (.6 x Earth to Moon distance) and the angle of the back-side of the moon visible is not compromised greatly. Lesser boost velocities are feasible but were not investigated in this note.

Conclusions

The application of a modest boost increment to a relay (Via the S-4-B) at or near translunar injection appears to be a desirable way of establishing a relay to communicate with the back-side of the moon. For a boost velocity of 160 ft/sec., approximately 150-130 degrees of the back-side of the moon will be visible to the relay during the lunar portion of the Apollo mission. The range between the relay and the moon will not exceed 0.6 of the moon distance from the earth to the moon.

Recommendations For Future Work

The analysis in this note is at best a first order approximation to the relay orbits. With regard to orbit determination, the following future studies are recommended.

1. Relay orbit determination on a computer program which solves the restricted three body problem.

2. Investigation of sensitivity of injection conditions to desirability of relay orbit.

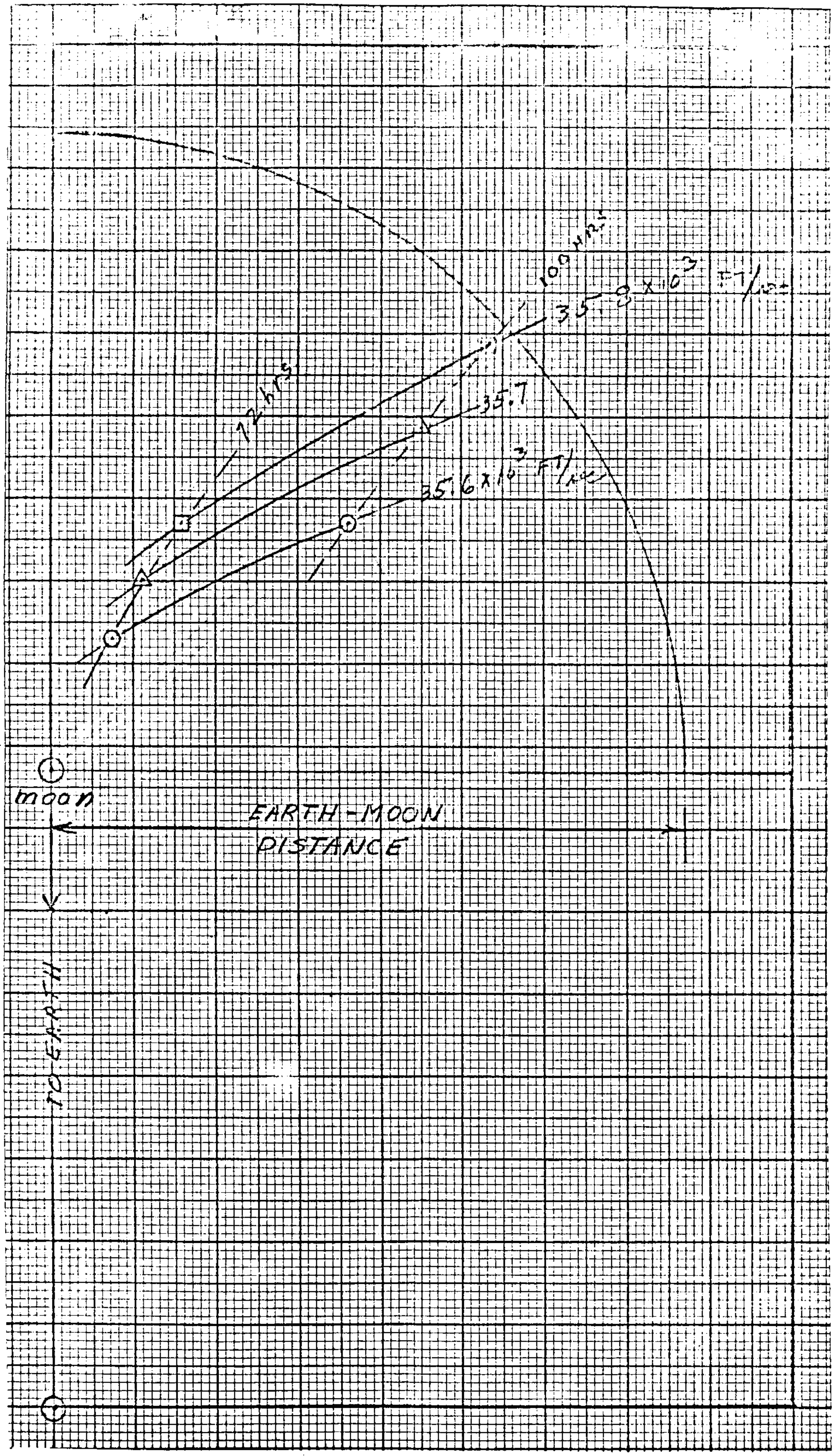


Figure 5.

Relay Trajectories Relative to the Moon

3. Comparison of boosts at translunar injection with boosts applied at other portions of the translunar orbit.

ADDENDUM

H. Epstein
18 April 1963

COMMUNICATIONS CAPABILITY OF UNSTABILIZED S-4-B SATELLITE RELAY SYSTEM

The desire to eliminate or at least minimize the stabilization problem for the S-4-B in the satellite relay mode of operation makes it desirable that the S-4-B employ omni-directional antennas. The capability in terms of bandwidth for such a system will be indicated below. Emphasis is placed on considerations involving the Apollo spacecraft and the S-4-B. In an earlier portion of this Apollo Note, it was indicated that the distance between the spacecraft and the S-4-B would be less than about 250,000 km for the desired operating period. For the sake of convenience, a scaling operation will be performed on the parameters selected in Apollo Note No. 35.

The primary parameters involved in the improvement of the range performance for an omni-antenna in the S-4-B are transmitter power, receiver noise figure, and Apollo spacecraft antenna gain. Restricting the D. C. power requirements to about 100 to 200 watts for the electronic equipment limits the transmitter RF power level to about 25 watts with amplitrons and 10 watts with cavity amplifiers (Apollo Note No. 18, Page 4). The noise figure with tunnel diodes being employed should be about 6 db (Apollo Note No. 18, Page 5). The 4-foot antenna in the Apollo spacecraft could be increased to about 5-foot (as suggested in the BellCom report for T. V. application) without undue complications with a resultant 2 db increase in antenna gain. The threshold sensitivity for the transponder to maintain lock 95% of the time with a 20 cps receiver bandwidth is about -154 dbm (JPL TM No. 33-26, DSIF Specification, Volume I, Page II. B-24 and II. B-25). A conservative design would require about a 6 db greater signal level. In addition, for a conservative design a 10 db allowance should be made for allowable omni-antenna gain, pointing inaccuracy of the directional antenna, transmitter degradation, and receiver degradation.

The following table indicates the bandwidth capabilities for the Bogie System, a maximum performance system, and a conservative system design. The latter two designs make use of an amplitron (power levels = 25 watts) and a low-noise tunnel diode receiving system (noise figure = 6 db). The performance is indicated for an omni-antenna and separations of 250,000 km for all systems.

Table 1.
Bandwidth Capability

	Bogie System	Maximum Performance	Conservative Design
Transmitter Power (dbm)	23	44	44
Improvement (db)		21	21
Directional Antenna Gain (db)	26.5	28.5	28.5
Improvement (db)		2	2
Noise Figure (db)	11	6	6
Improvement (db)		5	5
Conservative Performance Factor (db)	0	0	-16
Signal Strength (dbm)	-158	-130	-146
Relative Bandwidth Increase (db)	-4	+24	+8
Maximum Allowable Bandwidth (cps)	8	5000	125

Scaling laws such as those indicated above make the effect of other combinations of parameters rather simple to obtain. Personal past experience indicates that probably several hundred cycles per second of bandwidth could be expected.

To complete this picture, it is necessary that a more complete discussion of noise sources take place. For internal noise figures as typified by the above mentioned designs, only solar noise can present an external noise source which could markedly degrade the system performance. To be specific, the effective noise temperature of the sun at DSIF frequencies is about $80,000^{o}K$ for the quiet sun and $10,000,000^{o}K$ for the disturbed sun for suitably narrow beamwidth antennas when pointed at the sun. (These noise temperatures correspond to noise figures of about 24 and 45 db respectively). Effective solar temperatures for antennas, pointed at the sun, with antenna beamwidth which completely encompass the sun are reduced approximately by the ratio of the solid angle subtended by the antenna beamwidth to the solid angle subtended by the sun. This reduction factor is about 53 db for an omni-antenna and about 20 db for a five foot antenna at the S-Band DSIF frequency. It is interesting to note that an omni-antenna receiving system in the S-4-B would be relatively unaffected by solar noise while the directional antenna receiving system in the Apollo spacecraft would only be slightly affected by the quiet sun and very much affected by the disturbed sun (about 25 db reduction in bandwidth capability would result). As a matter of fact, when limited by solar noise alone the signal-to-noise is independent of receiving antenna gain until the point is reached that the antenna main-lobe solid angle no longer encompasses the entire sun (about .5 degrees in each dimension for circular beams). This situation can be alleviated by the design of antenna with low sidelobe levels in the direction of the sun if the main-lobe of the receiving system is not required to illuminate the sun. Where this limitation is a serious deterrent only improvements in the transmitter power level and antenna gain or making extremely small receiving antenna beamwidth can markedly improve the performance.

Conclusions

1. An information bandwidth in excess of 100 cps can be employed for separation distances between S-4-B and Apollo spacecraft of 250,000 km. This performance level is obtainable with an unstabilized antenna system on the S-4-B, and a D. C. power level from 100 - 200 watts.

2. Solar noise can place a severe limitation on attainable performance. This necessitates that directional antennas be designed with low side-lobe levels. Furthermore, from a communications viewpoint, it is highly desirable that the main-lobe of directional antenna systems not be required to illuminate the sun. This would place a restraint on desirable trajectories.

Section 7.

FAR-SIDE RELAY

L. Lustick/
C. Siska

Additional trajectory calculations were made on the far-side relay to see if boost conditions could be established which would allow voice communication with the CM/LEM. It is desired to have voice communication capabilities during the portion of the mission from deboost into lunar orbit to rendezvous between LEM and CM (a period of approximately 32 hours). It is particularly important to have voice communication at the time of deboost of CM into lunar orbit.

The ground rules specified by Mr. Fordyce allowed boosts as large as 1000 ft/sec. to be applied within the first seven hours following translunar injection. The range between the relay and CM consistent with voice communication was given as 40,000 nautical miles.

Method

Nominal translunar injection conditions were established which were approximately consistent with the arrival of the CM at perilune (100 n.m.) 72 hours after injection. The effect of perturbations in the velocity vector, both at translunar injection and approximately 7 hours after injection were examined. The locus of the position of the relay relative to the CM/SM at the time when the CM/SM pierces the LSOI was established. These Loci are shown in Figure 1. The elongated ellipse is for a boost at translunar injection of 1000 ft/sec. The different points on the locus correspond to different boost directions relative to the reference velocity vector as indicated in the upper left diagram in Figure 1. The other ellipse shown in Figure 1 corresponds to applying a boost of 1000 ft/sec. approximately 7.6 hours after translunar injection.

In lunar space, each point on the locus is traveling roughly in a 45 degree direction from lower left to upper right, and therefore, one can quickly estimate which points will penetrate the lunar sphere of influence.

Results

The trajectories of several points on the loci of Figure 1 were examined briefly in lunar space and at first glance, it appears that the positions around $\alpha = -90^{\circ}$ for the 7.6 hr. delayed boost are the most promising to fulfill the mission requirements.

Figure 2 shows the trajectory for the α = -90 boost in lunar space and also the reference lunar vehicle trajectory. The lunar vehicle enters the LSOI at 60 hours after translunar injection and arrives at perilune (for deboost into a circular orbit) approximately 12 hours later. Corresponding positions for the booster (Far-side Relay) are indicated. The perilune visibility limit shown in Figure 2, (i.e., the tangent to the lunar surface which passes through the perilune position) indicates that perilune is always visible to the booster position. Approximately thirty hours after lunar vehicle deboost, the Far-side Relay has approached the 40,000 n.mi. communications limit. Thus, it appears that the Far-side Relay will be within the voice communications limit for both lunar deboost and lunar rendezvous. Although it appears occultation by the moon occurs at 102 hours, this presents no problem since the trajectory can be shifted with slight changes in boost direction around $\alpha = -90^{\circ}$.

Far-side Relay trajectories going the other way around the moon (counter-clockwise), say for boosts slightly less than $\alpha = 90^{\circ}$, may also fulfill the mission requirement. This alternative procedure is yet to be investigated.

Conclusions

Assuming that a 1000 ft/sec. boost is available at approximately 7 hours after translunar injection, voice communications via the Far-side Relay appears feasible for both the lunar deboost and lunar rendezvous portions of the Apollo mission.

Future Tasks

1. Write a computer program based on the Egorov Model to facilitate the trajectory calculations so that a more complete evaluation of the far-side relay potential can be obtained.

2. Establish the nominal trajectory for the CM/SM more accurately. That is, what are translunar injection conditions that are consistent with a free return trajectory.

3. Investigate the effect of errors in the boost velocity on the far-side relay trajectory.

4. Determine expected orientation errors in the reference system at the time of boost and decide how the boost is to be executed.

5. Investigate the potential of the far-side relay as an aid to navigation.

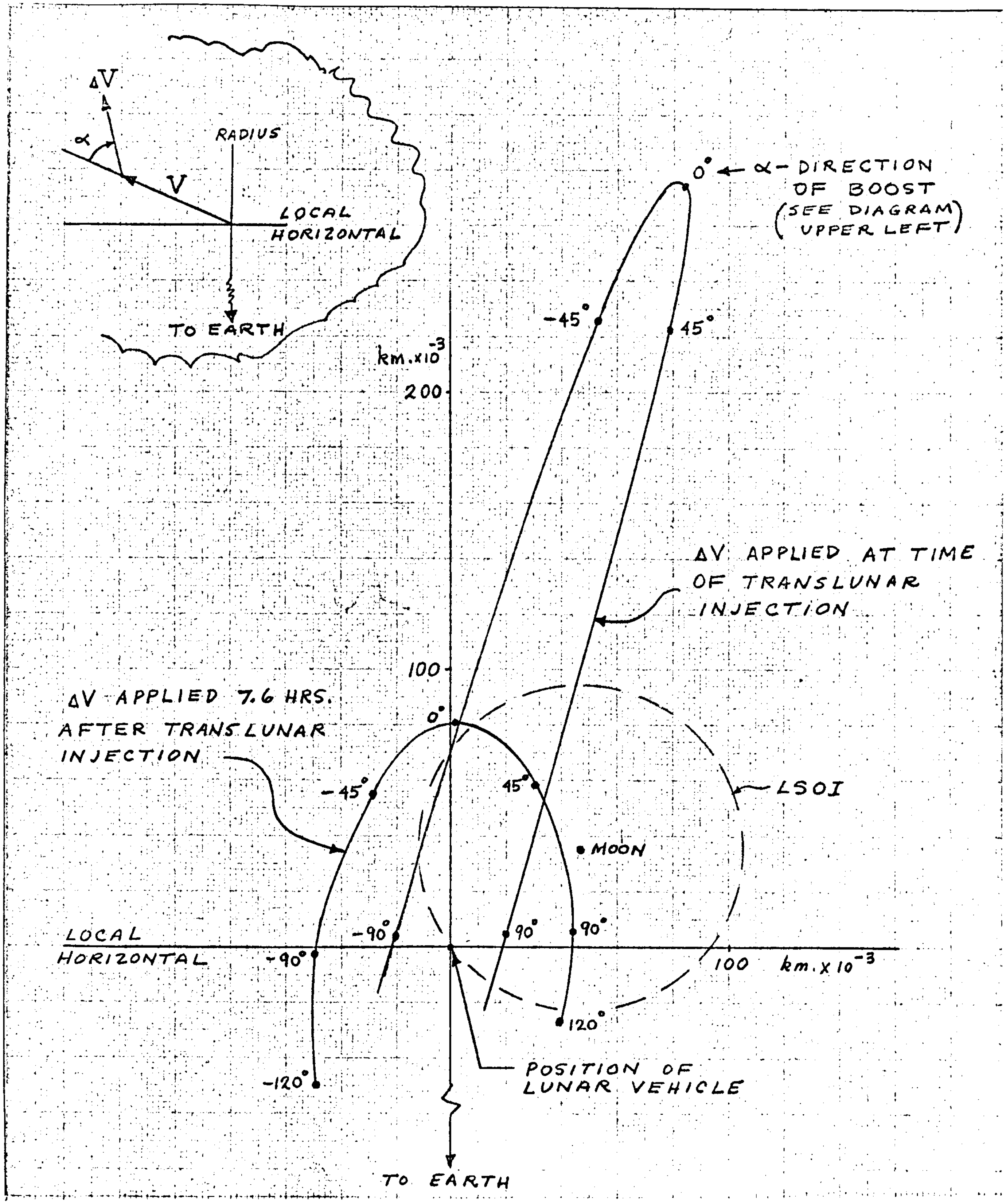


Figure 1. Relative Booster Positions At Time Lunar Vehicle Enters Lunar Sphere Of Influence - ΔV = 1000 ft/sec.

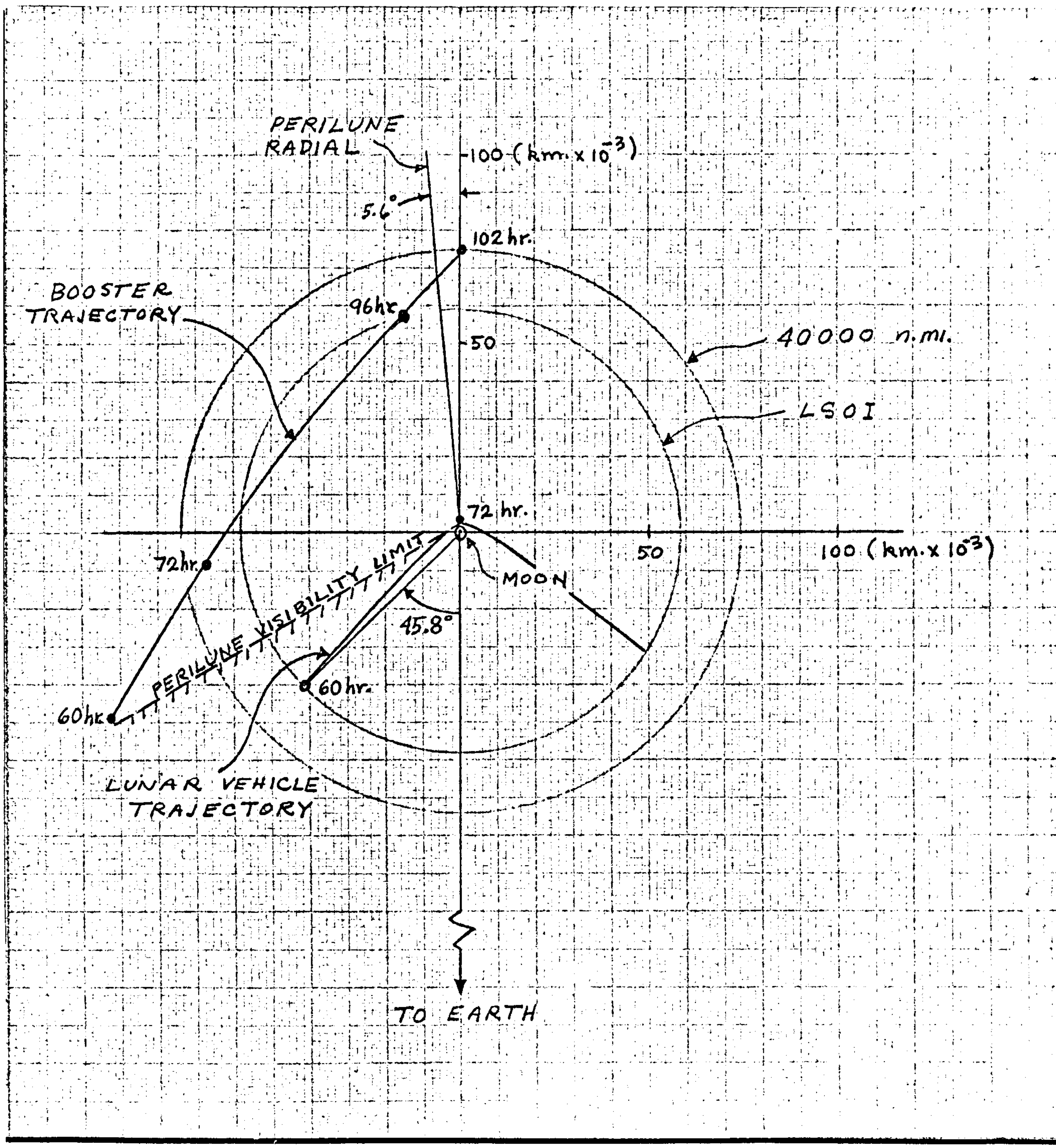


Figure 2. Booster Trajectory For $\Delta V = 1000$ ft/sec. and $\alpha = -90^{o}$

APOLLO NOTE NO. 90

C. Siska
6 August 1963

FURTHER EXAMINATION OF FAR-SIDE RELAY TRAJECTORIES

DISCUSSION

This note represents a continuation in the study of Far-side Relay trajectories as explored in Apollo Note No. 44 and Section 7 of Apollo Note No. 87.

There exists an interest for using the S IV booster as a voice communications relay between locations back of the Moon and the Earth during the times of lunar vehicle deboost from the translunar trajectory, and lunar rendezvous prior to Earth return. These periods of time occur approximately 72 and 100 hours, respectively, after translunar injection. A slant range limit of 40,000 n.mi. from the Moon has been adopted as consistent with the power requirement involved in the voice communication.

It is assumed that a velocity impulse of up to 1000 ft/sec. can be applied to the S IV booster at any time during a period of approximately 7 hours after translunar injection.

The feasibility of fulfilling the above-mentioned criteria is shown in Apollo Note No. 87, which illustrates a representative trajectory in the vicinity of the Moon.

In this note, the relation between boost velocity and direction, and the time of boost application is explored in a cursory manner, in order to indicate the operating region for these characteristics.

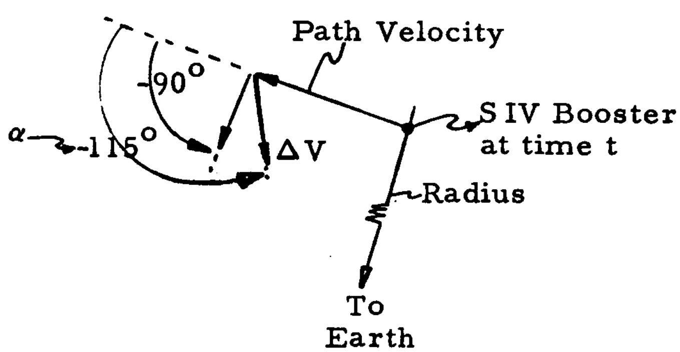
RESULTS

A graphical-analytic procedure has been used to determine the approximate lunar trajectories which appear in this note.

The particular combinations of operating characteristics which have been investigated are as follows:

t	ΔV	α
Time of Boost Application (hours after translunar injection)	Boost Velocity (ft/sec.)	Boost Direction Relative to Path Velocity
7.6	1,000	-120° to $+120^{\circ}$
7.6	500	-120° to $+120^{\circ}$
0	1,000	-120° to $+120^{\circ}$

It quickly became apparent that the range of boost directions, α, which might satisfy the mission requirement was approximately $-90^{\circ} < \alpha < -115^{\circ}$, as depicted in the following diagram.



To approximate the range of admissible values of α for each combination of ΔV, and t, the $\alpha = -90^{\circ}$ trajectory was computed for each case to represent the one limit, and then the other limit of α was obtained by searching for the trajectory which yielded perilune visibility at t = 72 hours. These two trajectories for each combination of ΔV, t are illustrated in Figures 1, 2, and 3.

The vertical axis of the co-ordinate system has been chosen to be the Earth-Moon line at t = 60 hours. This line is moving around the Moon in a counterclock-wise direction at a rate of 0.55^{o}/hour, and therefore, at some time part of the trajectory will not be visible to the Earth. Relative to the Earth, the path velocity is approximately 2 km/sec. so that the duration of occultation is approximately one-half hour.

Using Figures 1, 2, and 3, one can develop the operating region for given criteria as shown in Figure 4. Permissible values of ΔV and α for a given t lie within the region bounded by the specified t contour. This contour, as shown in Figure 4, consists of two segments; the left side is associated with the upper trajectory of Figures 1, 2, and 3 (leaving the 40,000 n. mi. circle at t = 100 hours), while the right side is for the lower trajectory (perilune visibility at t = 72 hours). A third segment which completes the contour is not shown and this would represent the situation when the t = 72 hour position lies on the 40,000 n. mi. circle.

Note that if the upper limit of ΔV is 1000 ft/sec., then applying the ΔV at t = 0 offers hardly any margin for error in thrust direction. Therefore, it appears preferable to apply ΔV sometime after translunar injection. However, thrust direction accuracy is expected to diminish with time because of gyro drift associated with the stable platform. Furthermore, evaporation of the residual fuel in the S IV booster may significantly lower the ΔV below the estimated value of 1000 ft/sec. for times after t = 0. These factors have not been given consideration up to the present time.

It can be noted that some combinations of ΔV and α yield counter-clock-wise lunar trajectories. However, none of these will simultaneously satisfy the criterion of observing the lunar vehicle perilune position at slant ranges of not more than 40,000 n. mi. for both t = 72 and t = 100 hours.

CONCLUDING REMARKS

A cursory analysis has shown that an operating region exists for the velocity impulse and direction for the S IV booster which will satisfy the Far-side Relay voice communication requirement.

A precise definition of the boost conditions for the Far-side Relay should involve the following considerations:

1. A more extensive set of data to provide a more accurate and detailed delineation of the boost operating region (such as illustrated in Figure 4). This data can be most efficiently collected by means of a computer program using an Egorov Model.

2. Examination of the velocity impulse and thrust direction accuracy available with time after translunar injection.

3. A final check on the selected design operating point using three-body trajectory equations.

To the above should probably be added the consideration of possible secondary missions of the Far-side Relay which may influence the particular reference trajectory chosen. For example, the Far-side Relay might be used as a navigation aid, together with an Earth-based computer, for lunar rendezvous steering commands to the LEM.

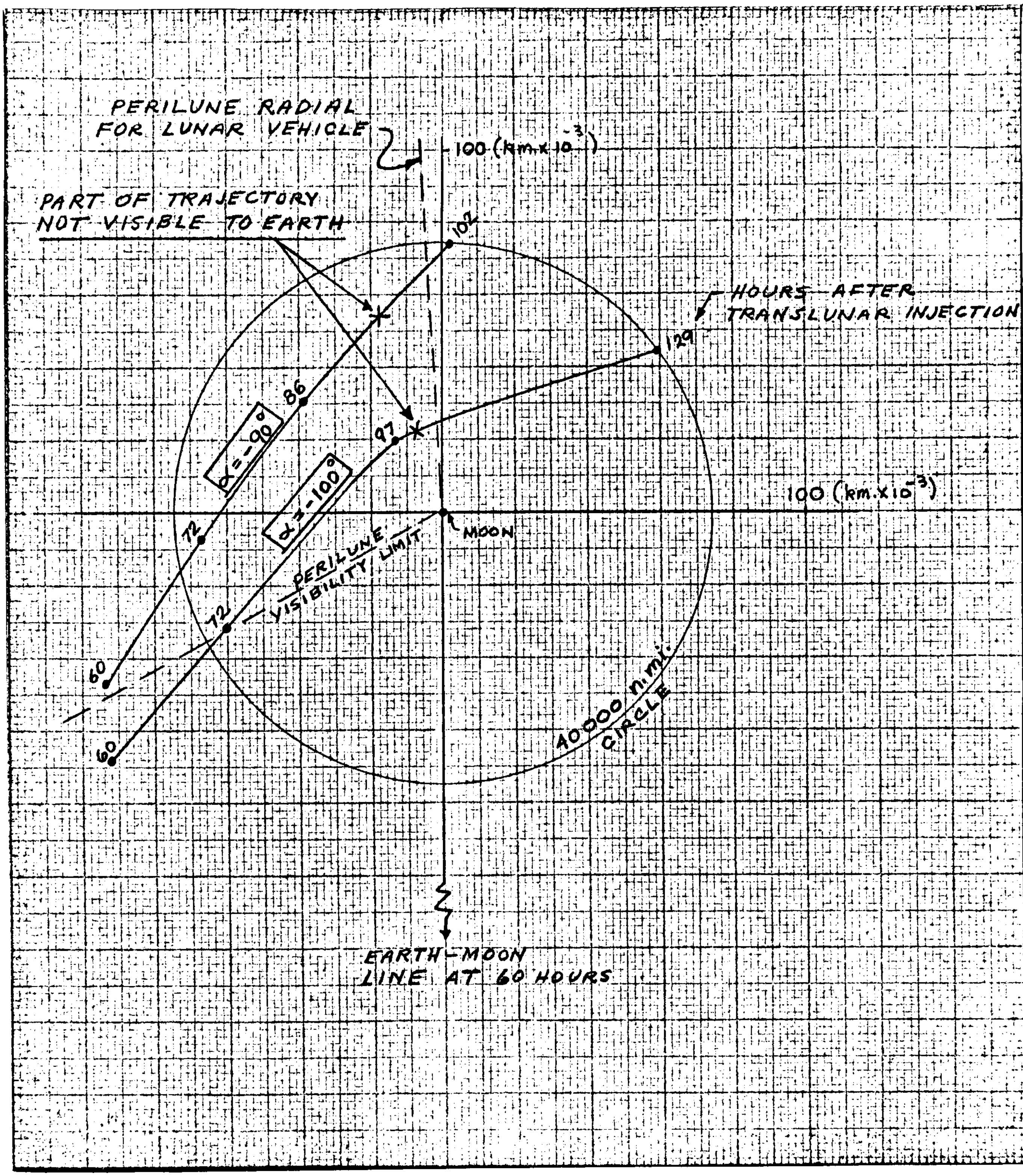


Figure 1. Far-side Relay Trajectories for ΔV = 1000 ft/sec. Applied 7.6 Hours After Translunar Injection.

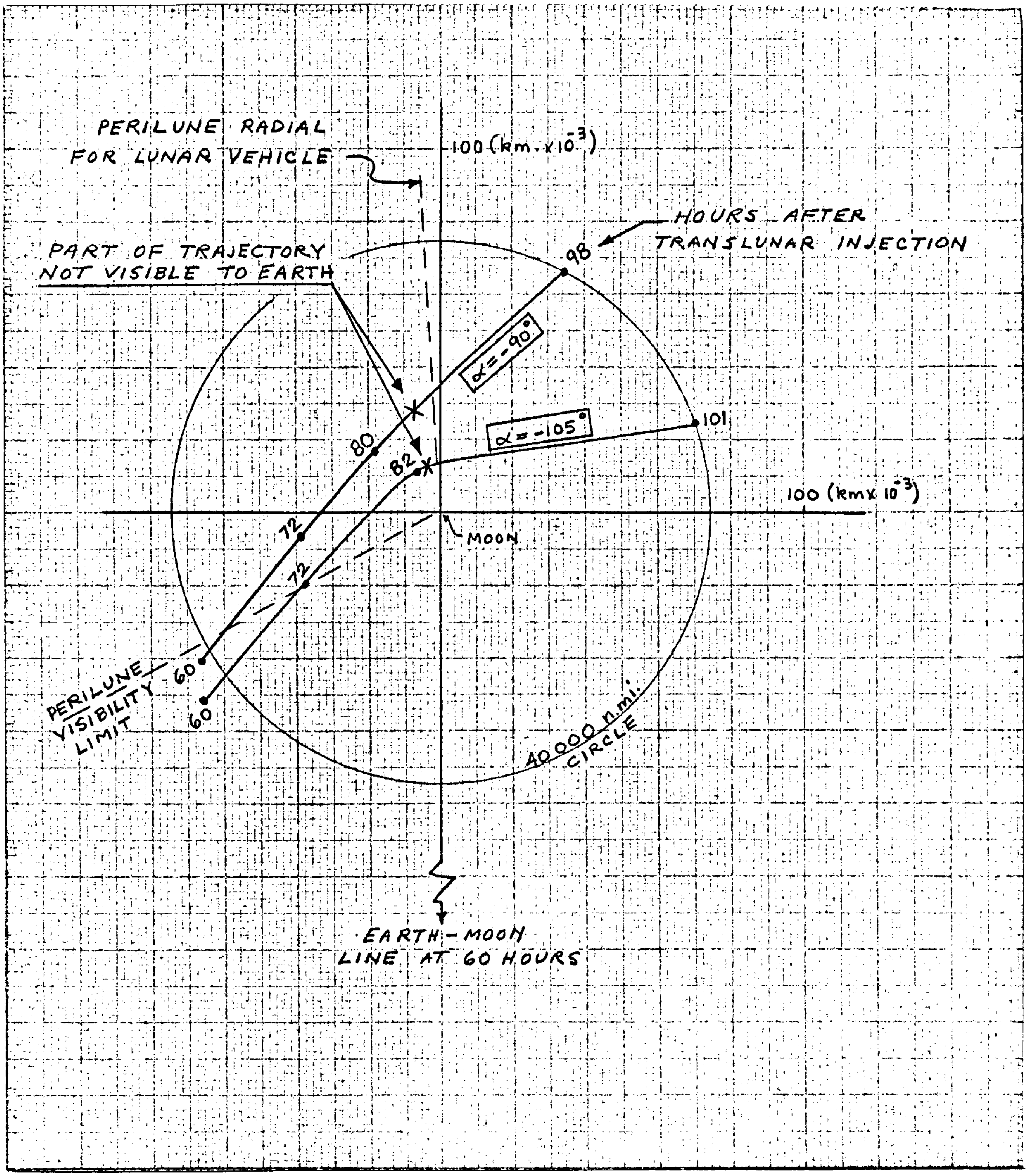


Figure 2. Far-side Relay Trajectories for $\Delta V = 500$ ft/sec. Applied 7.6 Hours After Translunar Injection.

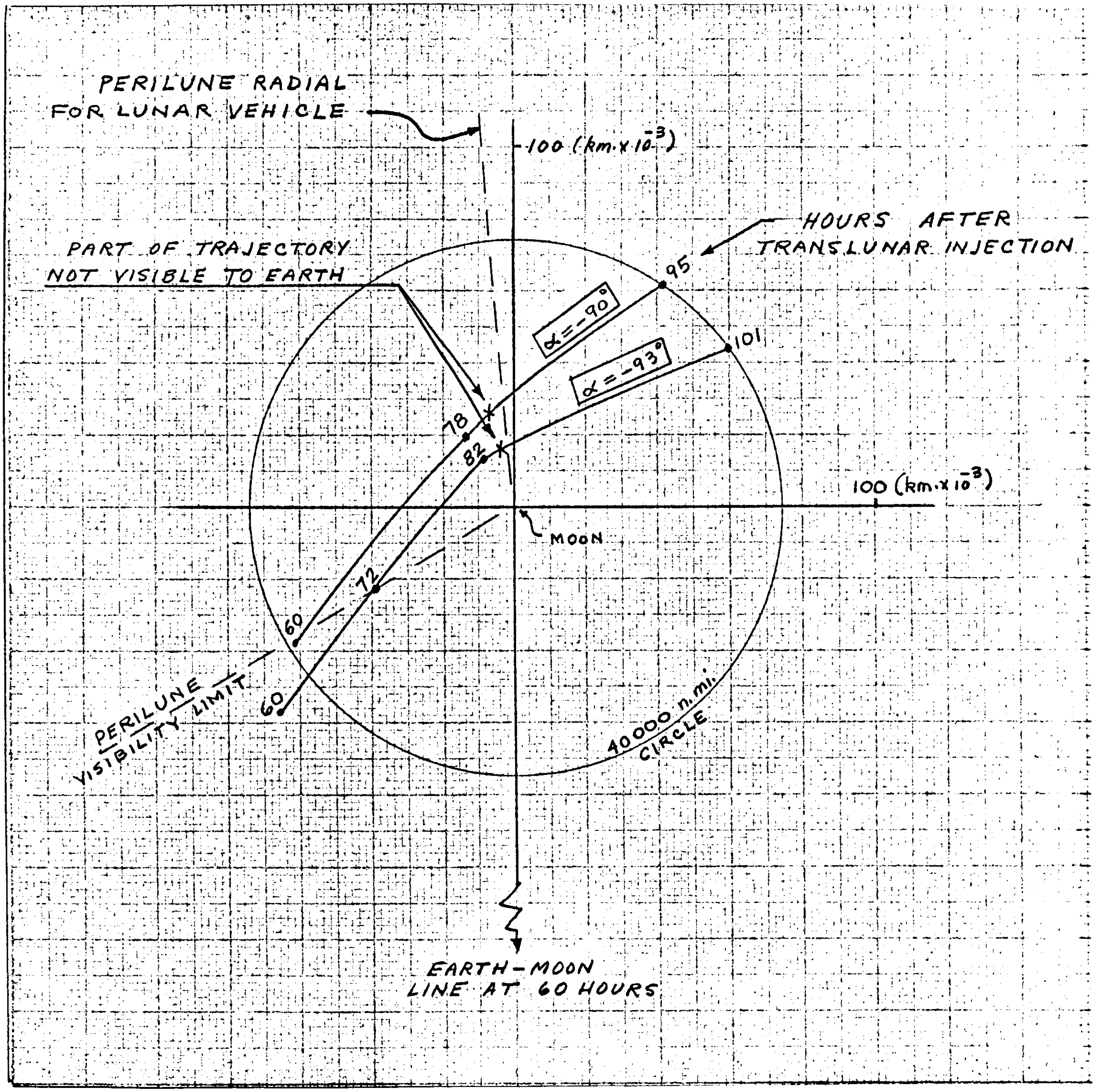


Figure 3. Far-side Relay Trajectories for ΔV = 1000 ft/sec. Applied at Time of Translunar Injection.

Limiting Criteria:

1. Lunar Vehicle Deboost Visible (t = 72 hours)
2. Lunar Rendezvous Visible (t = 100 hours and approx. same position as deboost)
3. Slant Range at Above Times ≤ 40,000 n. mi.

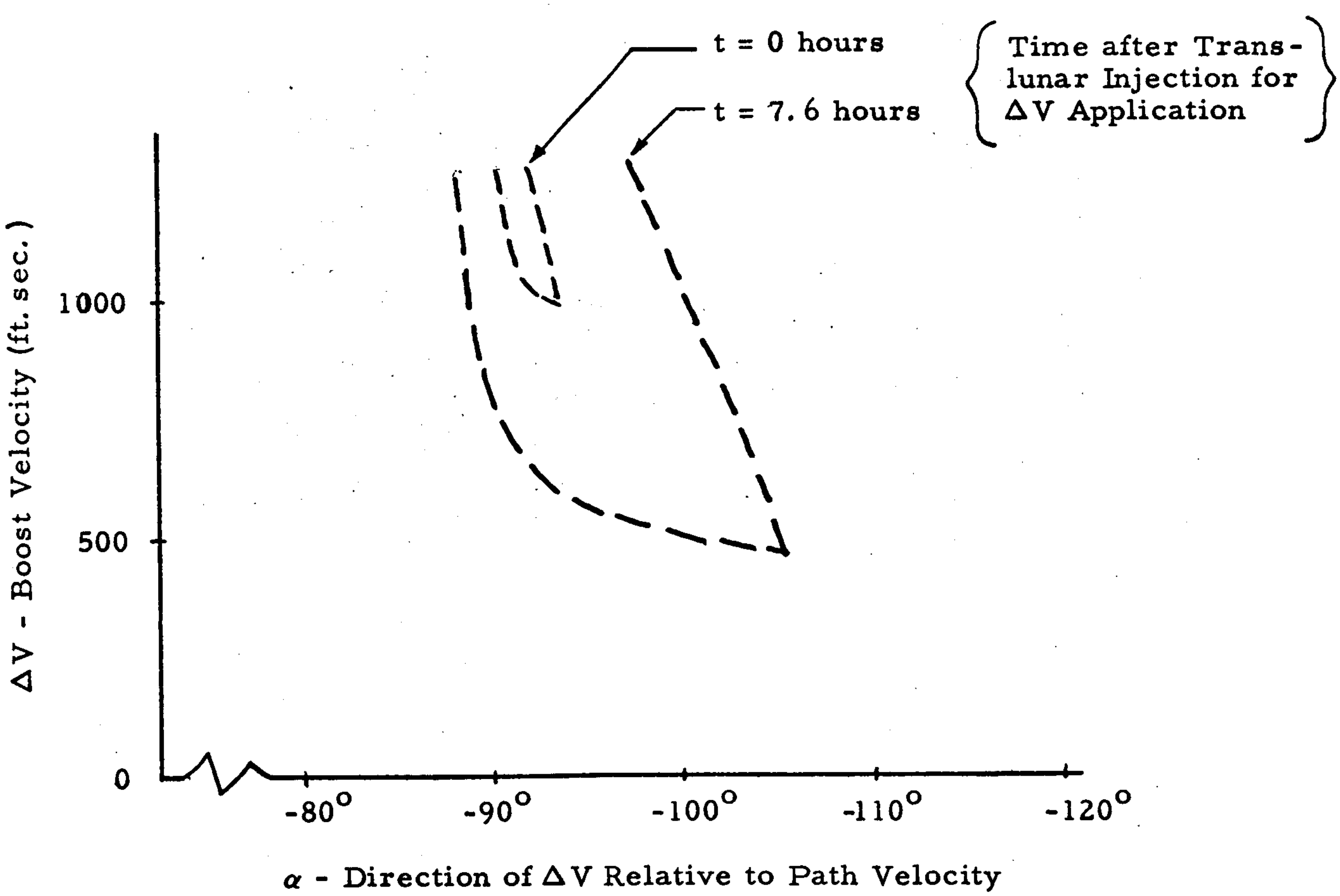


Figure 4. Approximate Operating Region for Far-side Relay Boost.

APOLLO NOTE NO. 97

C. Siska
20 August 1963

MINIMUM BOOST VELOCITY REQUIREMENT FOR FAR-SIDE RELAY

PURPOSE

This note presents data which augments the data appearing in Apollo Note No. 90.

RECAPITULATION

There is an interest in using the S-IV booster as a far-side relay to facilitate voice communications "back of the moon" to earth during the lunar deboost and lunar rendezvous operations. In principle, after the S-IV booster injects the lunar vehicle into a translunar orbit and is jettisoned, an additional boost can be applied to send the booster on its own translunar trajectory. To fulfill the far-side relay requirements, as presently defined, the S-IV booster must be within a slant range of 40,000 n. mi. from the lunar vehicle deboost position at a time approximately 72 and 100 hours after translunar injection (or, equivalently, S-IV booster jettison).

Apollo Note No. 90 indicates the operating region for the boost velocity, ΔV, and its direction α, relative to the path velocity, in order to fulfill far-side relay requirements when the ΔV is applied at 0 and 7.6 hours after translunar injection. Representative far-side relay trajectories are also shown in the note.

The present note examines the case when ΔV is applied at 4.15 hours after translunar injection and shows the resulting compilation of data.

RESULTS

Figures 1. and 2. show representative far-side relay trajectories in lunar space for ΔV values of 1000 and 700 ft/sec respectively. The direction of ΔV, denoted by α, is measured relative to the path velocity existing at 4.15 hours after translunar injection; $\alpha = 0$ indicates that ΔV is directed along the path velocity and negative α values decrease the angle the path velocity makes with the local horizontal.

By combining results such as shown in Figures 1. and 2., one can develop limiting contours in the ΔV, α plane as shown in Figure 3. Combinations of ΔV and α which satisfy the indicated limit criteria lie within a specified contour. The left hand side of each contour is dictated by the criterion that the $t = 100$ hour positions lie on the 40,000 n. mi. circle (see Figures 1. and 2.), while the right hand side is associated with seeing the lunar vehicle deboost position at $t = 72$ hours.

Now the time when ΔV is applied will have an influence on the magnitude of ΔV which is available at that time, because of fuel "boil-off".

Thus, it would appear that from the consideration of boost velocity availability and boost requirements, there exists an upper limit to the time for applying the boost. The minimum ΔV required, as a function of time, can be obtained from Figure 3. and the resulting curve is shown in Figure 4.

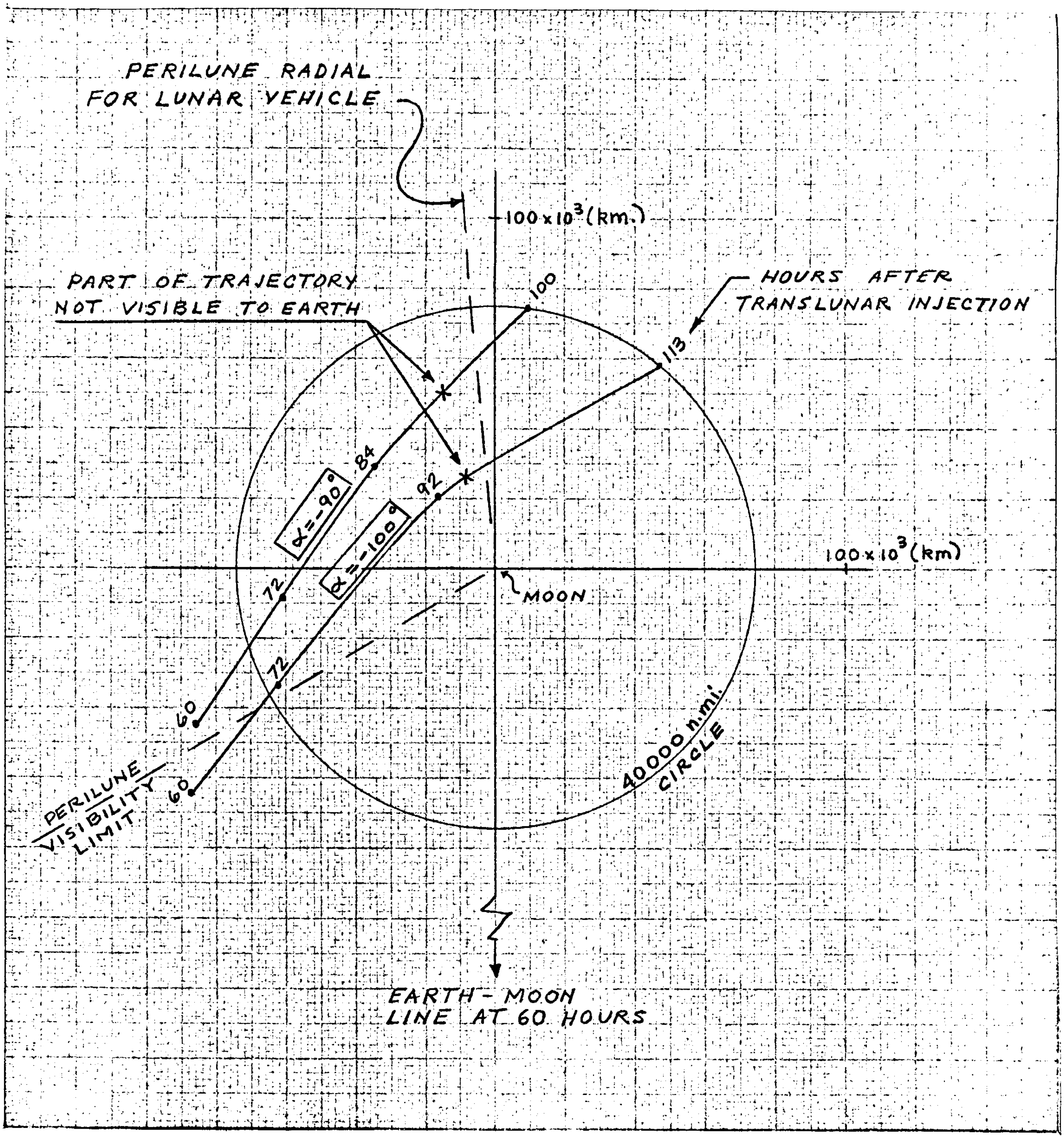


Figure 1. Far-side Relay Trajectories for ΔV = 1000 ft/sec. Applied 4.15 Hours After Translunar Injection.

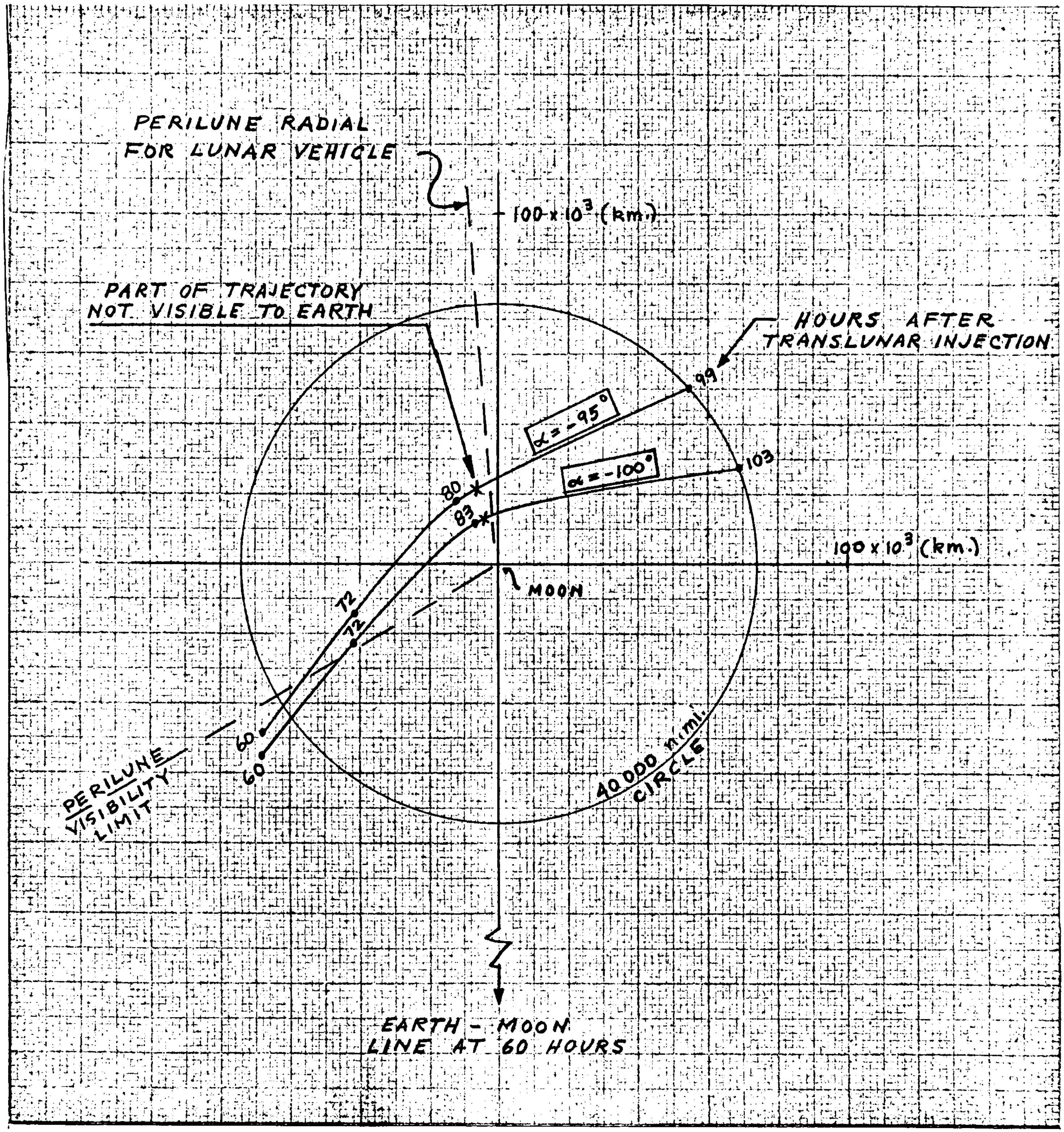


Figure 2. Far-side Relay Trajectories for ΔV = 700 ft/sec. Applied 4.15 Hours After Translunar Injection.

Limiting Criteria:

1. Lunar vehicle deboost visible (t = 72 hours)
2. Lunar rendezvous visible (t = 100 hours)
3. Slant range at above times $\leq$ 40,000 n. mi.

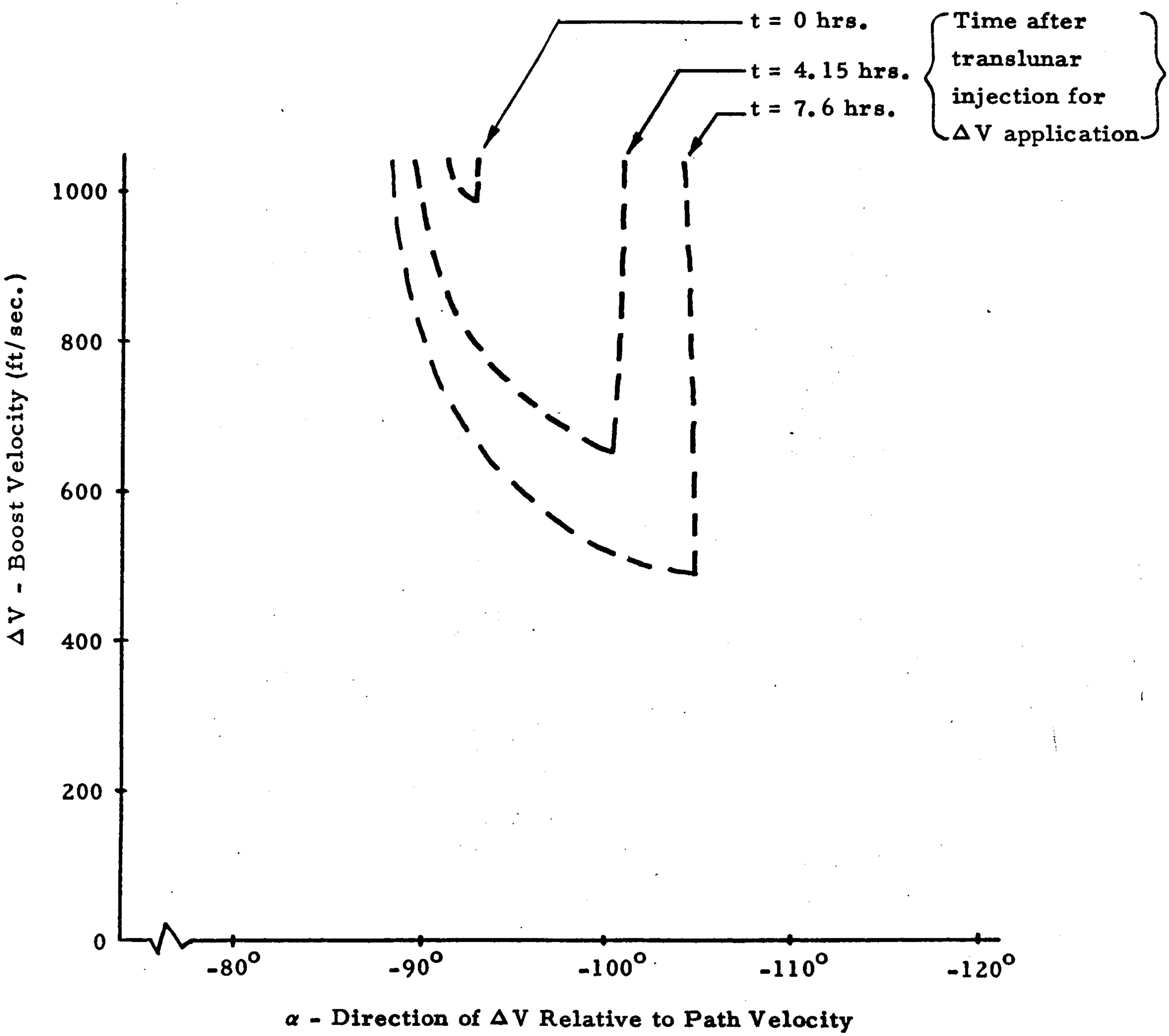


Figure 3. Approximate Operating Region for Far-side Relay Boost

Limiting Criteria:

1. Lunar vehicle deboost visible (t = 72 hours)
2. Lunar rendezvous visible (t = 100 hours)
3. Slant range at above times $\leq$ 40, 000 n. mi.

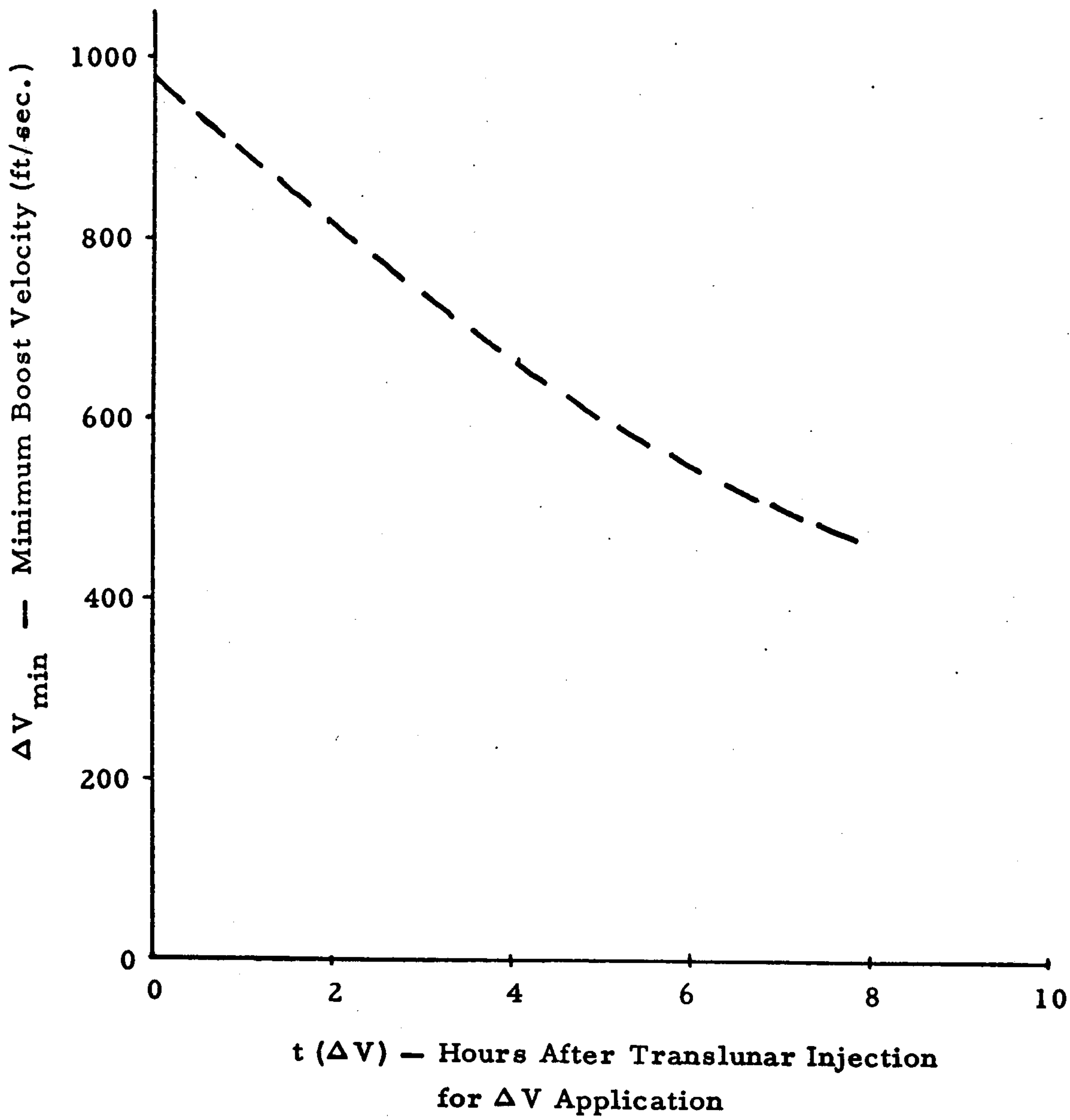


Figure 4. Minimum Boost Velocity Requirement for Far-side Relay

PART III

CM/SM ABORT GUIDANCE

PART III: CM/SM ABORT GUIDANCE

This section is concerned primarily with the problem of returning the CM/SM to Earth, using DSIF or MSFN assistance, in the situation where the CM/SM guidance system is severely crippled. It is shown that if the astronauts are capable of functioning, a safe return can be accomplished even though the on-board guidance system has failed and the main engine is locked hard-over.

Apollo Note No. 33, A Method of Manual Thrust Control During Boosts* describes a technique for thrust control using a star for orientation. Apollo Note No. 38 examines Spin Stabilization for Altitude Control During Boosts* and shows that use of spin stabilization provides control of thrust direction even with the main engine locked hard-over.

Next, Note 47, Efficient Boosting with Low Thrusts* investigates the possibility of return-to-Earth with the main engine out of commission, through the use of the attitude control jets. It is found that the time required for this kind of operation is too great for it to be practical in returning the CM from an orbit about the Moon, but that it may be useful in an abort when the main engine fails during deboost into lunar orbit.

Apollo Note No. 49, Pre-Boost Attitude Control* examines the likelihood that sufficient attitude control will exist for spin stabilization of the vehicle prior to boost, and describes the spin stabilization procedure.

Apollo Note No. 63, DSIF Capability on Trans-Earth Trajectory* 8/ describes a procedure for safe zero-lift return-to-Earth using the ground system for navigation, and stars and spin stabilization for guidance. It is shown that the proper trajectory for zero-lift re-entry can be achieved eight hours before re-entry.

8/ Employs results of Bissett-Berman Corporation Apollo Note No. 26. Error Analysis for Return-To-Earth Trajectories, Part 3.

APOLLO NOTE NO. 33

G. F. Floyd
19 March 1963

A METHOD OF MANUAL THRUST CONTROL DURING BOOSTS

After the main engine thrust axis has been pointed at the commanded star, there is still the problem of center of gravity location uncertainties. With the exact center of gravity location unknown, spacecraft body torques in roll pitch and yaw will result in angular acceleration of the thrust direction away from the commanded direction. Before considering the general three-dimensional case, we will consider only the case of a single plane (yaw or pitch but no roll). The definition of quantities is given in Figure 1 where all quantities are the actual quantities, even though the lack of center of gravity location will mean that we will not know where the actual vehicle axis is.

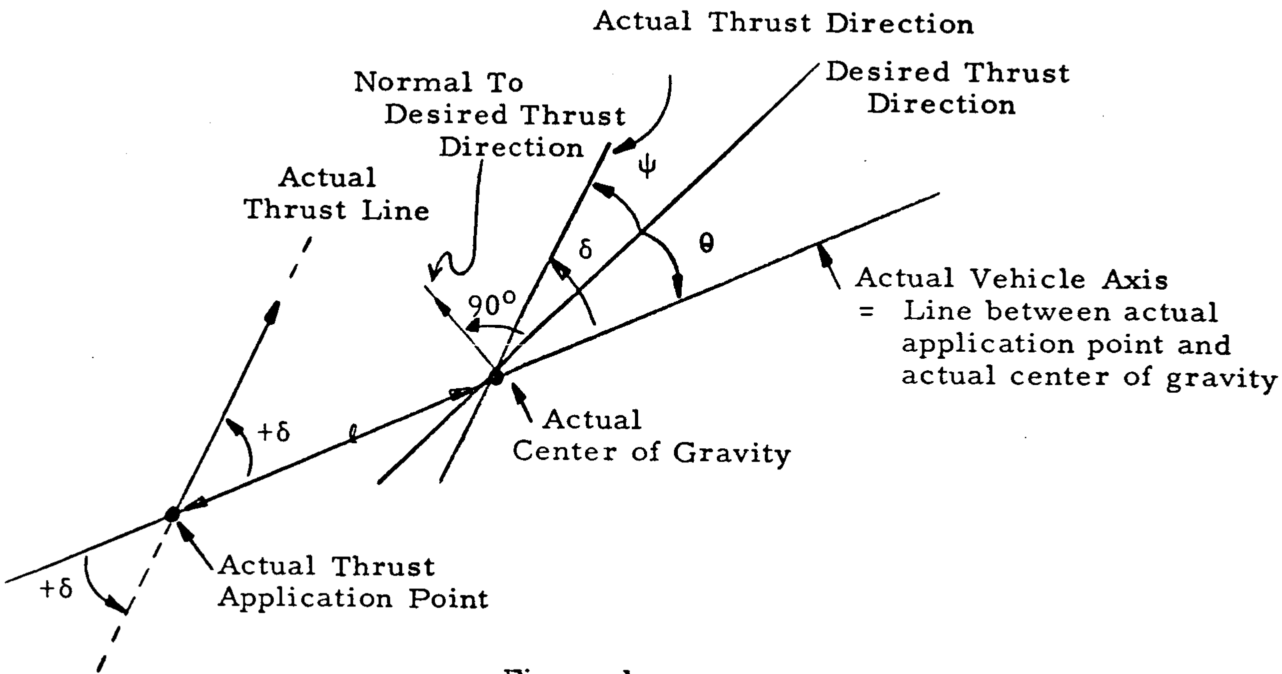


Figure 1.
Actual Motion Diagram

Let

$$\ell \triangleq \text{distance from actual thrust application point to actual center of gravity}$$

$$J \triangleq \text{actual moment of inertia}$$

$$m \triangleq \text{actual mass} \tag{1}$$

$$F \triangleq \text{actual thrust}$$

Then,

$$\dot{V}_n = \frac{F}{m} (\delta - \theta) \tag{2}$$

$$\ddot{\theta} = \frac{F\ell}{J} \delta \tag{3}$$

Now assume the rocket engine is held at a fixed gimbal angle so,

$$\delta = \delta_o \tag{4}$$

and initially the vehicle attitude rate is zero so,

$$\dot{\theta}_o = 0 \tag{5}$$

Also, at the start of a boost:

$$V_{n_o} = 0 \tag{6}$$

Using these conditions in 2 and 3 we have,

$$\theta = \theta_o + \frac{1}{2} \frac{F\ell}{J} \delta_o t^2 \tag{7}$$

and

$$V_n = \frac{F}{m} \left(\delta_o t - \theta_o t - \frac{1}{6} \frac{F\ell}{J} \delta_o t^2 \right) \tag{8}$$

So combining (7) and (8):

$$V_n = \frac{F}{m} t \left(\left(\delta_o - \theta_o \right) - \frac{1}{3} \left(\theta - \theta_o \right) \right) \tag{9}$$

Now with reference to Figure 1 and Equation (9), redraw Figure 1 as Figure 2.

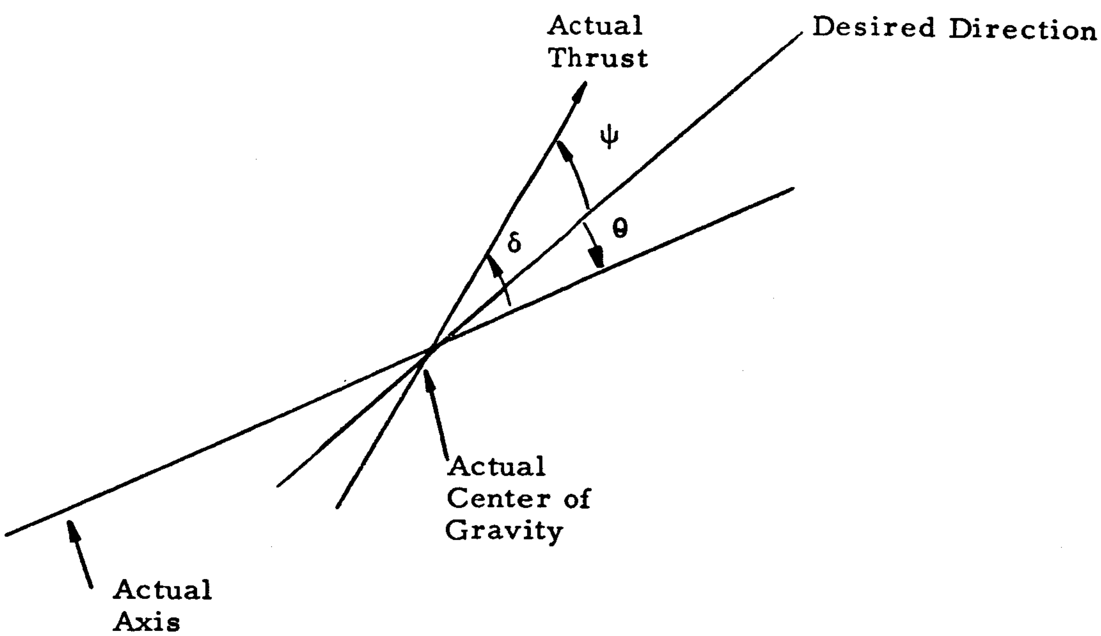


Figure 2.

Let

$\psi \triangleq$ angle between actual thrust and desired direction

$= \delta - \theta$

$\Delta\theta \triangleq$ change in actual axis = change in assumed axis (10)

$= \theta - \theta_o = \frac{1}{2} \frac{F}{J} \delta_o t^2$

With these definitions (9) becomes:

$$\boxed{V_n = \frac{F}{m} t \left(\psi_o - \frac{1}{3} \Delta\theta \right)} \tag{11}$$

From (11) we see that

$$\boxed{V_n = 0 \text{ when } \Delta\theta = 3\psi_o} \tag{12}$$

Now in the actual case we know the thrust direction with respect to some vehicle body axis (not necessarily through the unknown actual center of gravity), since the engine gimbal angle is known and the thrust alignment with respect to the physical nozzle can be accurately measured during preflight tests. Thus with reference to a known vehicle axis the thrust direction is known. Similarly, since the stars are visible this known axis can be accurately pointed at the star. Then with reference to Figure 2, ψ is known. Assume a ring and bead-like sight with the bead initially pointed at the star and the ring made three times the known ψ angle. Then if the astronaut stops thrusting when the star crosses the ring, V_n will be zero (see Equation 12). He should then redirect the bead at the star and repeat the thrust application, each time shutting off the engine when the star reaches the ring. For this adjustment we can calculate the thrust time deviation (t_o) of each pulse by using 12 to 10°. Thus,

$$\Delta \theta = \quad 3\,\psi_o = \frac{1}{2}\ \frac{F\ell}{J}\ \delta_o\, t_o^{\,2}$$

$$\boxed{t_o = \sqrt{\frac{6\,\psi_o\, J}{F\ell\,\delta_o}}}$$

Now (11) is a correct formula but to get the cancellation ψ and $\Delta\theta$ must be of the same sign, thus, we must make sure that the actual center of gravity a priori known with the correct sense, otherwise the V_n will increase rather than go through zero.

To see this more clearly and also discuss the modifications which must be made to the three dimensional case, consider Figure 3.

The figure is drawn in the plane normal to the desired thrust direction. Prior to firing,the spacecraft is aimed so the star is in the center of the inner ring. The outer ring is concentric with the inner end of a radius equal to three times the distance from the thrust direction (shown with a cross) to the star (Equation 12). The thrust must be to the same side of all possible center of gravity location in order to make the signs in (11) correct for cancellation. Since Figure 3 assumes to have no roll component, the line between the

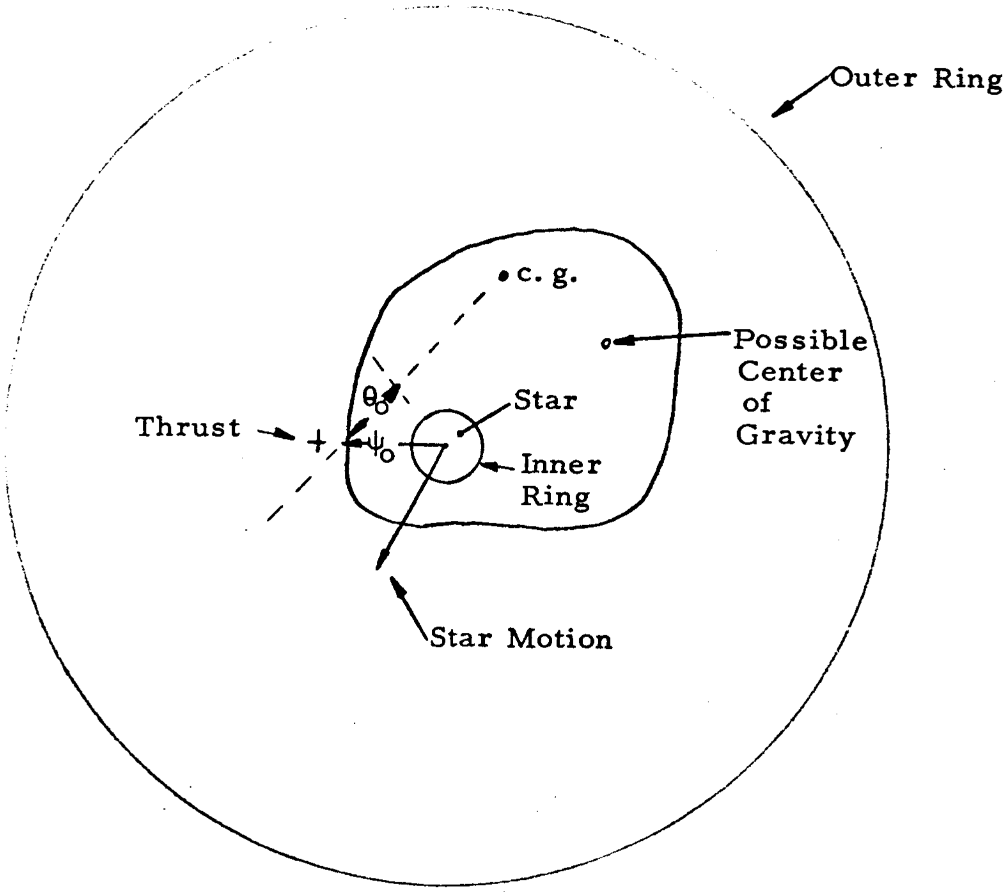


Figure 3.

thrust and actual center of gravity determines the direction of motion. Thus, since the thrust star and actual center of gravity are not in a line, the star does not pass through the thrust direction during its travel to the outer sight ring. Calling ϕ_o the angle between the thrust and the projection of the star on the thrust center of gravity line, we see that the radius of the outer circle should be 3 ϕ_o rather than 3 ψ_o where ψ_o is the angle between the thrust and the star. However, not knowing where the center of gravity is means we don't know ϕ_o so can't adjust the outer ring radius correctly. However, the fact that the star does not move through the thrust direction means that if on the next boost the pilot will readjust the thrust direction working down toward where the star moved, he will make $\phi_o \rightarrow \psi_o$, and when the star moves through the thrust direction $\phi_o = \psi_o$ and (12) will be satisfied when the star crosses the outer ring.

Thus by changing the engine gimbal angle and indicating the thrust direction in the sight by a cross, in successive boosts the pilot can get the star to move through the thrust direction mark, and so make $V_n = 0$ for each boost.

After the thrust axis, star and center of gravity are in line, then V_n for each pulse is zero and to increase the duration, hence magnitude, of each pulse the thrust can be brought closer to the center of gravity by moving the thrust line towards the star. This will require changing the diameter of the outer ring since by (12) it must be kept three times the initial thrust - star angle.

The method will not work with roll torques. However, the presence and direction of roll torques will be evident to the astronaut by the motion of the star relative to the sight since he will see the roll rates build up.

While this method appears possible in principle, it seems overly complicated and the roll problem will make it hard to execute.

APOLLO NOTE NO. 38

G. F. Floyd
4 April 1963

SPIN STABILIZATION FOR ATTITUDE CONTROL DURING BOOSTS

This note investigates the feasibility of spin stabilization during spacecraft maneuvers as a back-up in case the normal vehicle autopilot system fails. The symbols to be used in the analysis are defined in Figure 1.

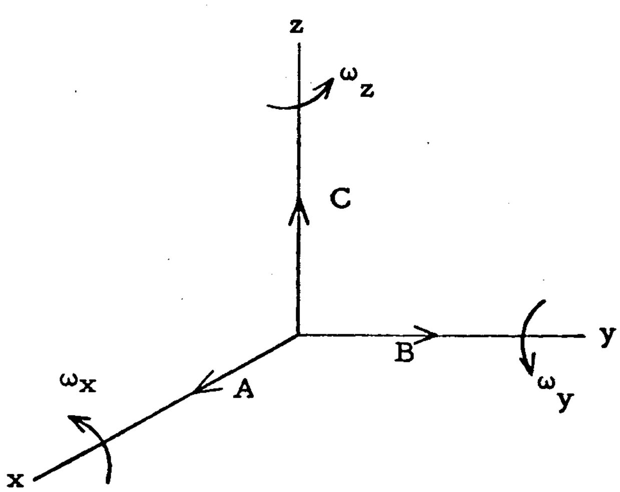


Figure 1.

The xyz is fixed to the spacecraft with z axis taken as the longitudinal axis of symmetry hence is also the desired thrust direction. The moments of inertia are A, B, and C along x, y, and z, respectively with x, y defined as

$$A > B > C \tag{1}$$

and from symmetry of the spacecraft

$$A \approx B \tag{2}$$

We will assume that the thrust is along z but because of c. g. shifts, that a resultant torque M_y is applied about the y axis which from (1) is the intermediate moment of inertia. The reason for taking the torque about (y) is that this is the unstable axis for steady spin. The equation of motion in vehicle fixed axes are:

$$\overline{M} = \overline{y}\, M_y = \dot{\overline{H}} = \left.\frac{d\overline{H}}{dt}\right|_{xyz} + \overline{\omega}_{xyz} \times \overline{H} \tag{3}$$

with

$$\overline{H} = \overline{x}\, A\omega_x + \overline{y}\, B\omega_y + \overline{z}\, C\omega_z \tag{4}$$

and

$$\overline{\omega}_{xyz} = \overline{x}\, \omega_x + \overline{y}\, \omega_y + \overline{z}\, \omega_z \tag{5}$$

Using (4) and (5) in (3) and equating components we get the usual Euler equations:

$$\boxed{\begin{aligned} 0 &= A\, \dot{\omega}_x + (C - B)\, \omega_y \omega_z \\ M_y &= B\, \dot{\omega}_y + (A - C)\, \omega_x \omega_z \\ 0 &= C\, \dot{\omega}_z + (B - A)\, \omega_x \omega_y \end{aligned}} \tag{6}$$

In this method the vehicle would first be spun about the z axis prior to the initiating of thrust, hence the initial conditions at the beginning of the thrusting are

$$\boxed{\begin{aligned} \omega_{x_o} &= \omega_{y_o} = 0 \\ \omega_z &= \omega_{z_o} \end{aligned}} \tag{7}$$

The solution of (6) gives the vehicle body rates in the body fixed coordinate system x, y, z, and to judge the feasibility of the spin stabilization it is necessary to calculate the motion with respect to inertial coordinates.

We will use the initial orientation of the xyz as this inertial system, (i. e. the x_o, y_o, z_o system) and instead of using Euler angles will work with the direction cosines. Then since we will initially align $\bar{z}$ with the desired boost direction we will use $\overline{z_o}$ as the correct thrust direction and be primarily interested in the angle between $\bar{z}$ and $\overline{z_o}$. Thus we define the set of direction cosines as:

$$\boxed{\begin{aligned} C_1 &\triangleq \bar{x} \cdot \bar{z}_o \\ C_2 &\triangleq \bar{y} \cdot \bar{z}_o \\ C_3 &\triangleq \bar{z} \cdot \bar{z}_o \end{aligned}} \tag{8}$$

Where from the definition we have

$$\boxed{\begin{aligned} C_{10} &= C_{20} = 0 \\ C_{30} &= 1 \end{aligned}} \tag{9}$$

and the relation between the C's:

$$C_1^2 + C_2^2 + C_3^2 = 1 \tag{10}$$

Differentiating (8) and using the fact that $\bar{x}$, $\bar{y}$, $\bar{z}$ are unit vectors we have with (8)

$$\begin{aligned} \dot{C}_1 &= \dot{\bar{x}} \cdot \bar{z}_o = (\bar{\omega}_{xyz} \times \bar{x}) \cdot \bar{z}_o = (\bar{y}\,\omega_z - \bar{z}\,\omega_y) \cdot \bar{z}_o \\ &= \omega_z (\bar{y} \cdot \bar{z}_o) - \omega_y (\bar{z} \cdot \bar{z}_o) = C_2\,\omega_z - C_3\,\omega_y \end{aligned} \tag{11}$$

Repeating this procedure for $\dot{C}_2$ and $\dot{C}_3$ we get the set:

$$\boxed{\begin{aligned} \dot{C}_1 &= C_2\,\omega_z - C_3\,\omega_y \\ \dot{C}_2 &= C_3\,\omega_x - C_1\,\omega_z \\ \dot{C}_3 &= C_1\,\omega_y - C_2\,\omega_x \end{aligned}} \tag{12}$$

In the general (and actual) case of unequal moments of inertia, this is about all that can be done analytically and for computer solutions the forms shown in boxes are easy to program.

Analytic Solution When A = B

In the case where A = B (which can never occur for an actual body), (6) reduces to the simple set:

$$\begin{aligned} 0 &= A\dot{\omega}_x - (A - C)\ \omega_z\,\omega_y \\ M_y &= A\,\dot{\omega}_y + (A - C)\,\omega_z\,\omega_x \\ 0 &= C\,\dot{\omega}_z \end{aligned} \tag{13}$$

Thus from the third we have,

$$\omega_z = \omega_{z_o}$$

and with the definitions

$$\boxed{\begin{aligned} \omega_n &\triangleq \left(1 - \frac{C}{A}\right)\omega_{z_o} \\ \omega_t &\triangleq \frac{M_y}{A\,\omega_n} \end{aligned}} \tag{14}$$

the first two of (13) becomes:

$$\begin{aligned} 0 &= \dot{\omega}_x - \omega_n\,\omega_y \\ \omega_n\,\omega_t &= \dot{\omega}_y + \omega_n\,\omega_x \end{aligned} \tag{15}$$

The solution of (15) with the initial conditions of (7) are then easily found as:

$$\boxed{\begin{aligned} \omega_x &= \omega_t \;(1 - \cos \omega_n t) \\ \omega_y &= \omega_t \sin \omega_n t \\ \omega_z &= \omega_{z_o} \end{aligned}} \qquad (16)$$

Since the C's are direction cosines and the angle we are interested in is the angle between $\overline{z}$ and $\overline{z}_o$, it is now convenient to define:

$$\begin{aligned} C_3 &\triangleq \cos \theta \\ C_2 &\triangleq \sin \theta \sin \phi \\ C_1 &\triangleq - \sin \theta \cos \phi \end{aligned} \qquad (17)$$

Where the association of cosine ϕ with C_1 and sin ϕ with $-C_2$ follows from the nature of C_1 and C_2 near $t = 0$ as defined by (12) and (16). In particular, on the basis of the behavior of the C's near zero as defined by (9), (12) and (16) we find easily that

$$\boxed{\theta_o = \dot{\theta}_o = \phi_o = 0} \qquad (18)$$

Using (17) in expression for $\dot{C}_3$ in (12) we get

$$- \dot{\theta} \sin \theta = - \omega_y \sin \theta \cos \phi - \omega_x \sin \theta \sin \phi$$

So $$\boxed{\dot{\theta} = \omega_x \sin \phi + \omega_y \cos \phi} \qquad (19)$$

Then solving (17) for ϕ, differentiating and using (12):

$$\dot{\phi} = \frac{d}{dt}\left[\tan^{-1}\left(\frac{C_2}{-C_1}\right)\right] = \frac{\dfrac{C_2\dot{C}_1 - C_1\dot{C}_2}{C_1^2}}{1 + \dfrac{C_2^2}{C_1^2}} = \frac{C_2(C_2\omega_z - C_3\omega_y) - C_1(C_3\omega_x - C_1\omega_z)}{C_1^2 + C_2^2}$$

$$= \omega_z - \left[\frac{\omega_y \sin\theta\cos\theta\sin\phi - \omega_x\sin\theta\cos\theta\cos\phi}{\sin^2\theta}\right]$$

So

$$\dot{\phi} = \omega_z - \left(\frac{\omega_y \sin\phi - \omega_x\cos\phi}{\tan\theta}\right) \tag{20}$$

Equation (19) and (20) are true in general and now using the solution (16) for the case of A = B we have

$$\dot{\theta} = \omega_t(1 - \cos\omega_n t)\sin\phi + (\omega_t\sin\omega_n t)\cos\phi$$

$$= 2\omega_t\sin^2\frac{\omega_n}{2}t\sin\phi + 2\omega_t\sin\frac{\omega_n}{2}t\cos\frac{\omega_n}{2}t\cos\phi$$

$$\dot{\theta} = 2\omega_t\sin\frac{\omega_n}{2}t\cos\left(\phi - \frac{\omega_n}{2}t\right) \tag{21}$$

$$\dot{\phi} = \omega_{z_o} - \left[\frac{\omega_t\sin\omega_n t\sin\phi - \omega_t(1 - \cos\omega_n t)\cos\phi}{\tan\theta}\right]$$

$$= \omega_{z_o} - \left[\frac{2\omega_t\sin\frac{\omega_n}{2}t\cos\frac{\omega_n}{2}t\sin\phi - 2\omega_t\sin^2\frac{\omega_n}{2}t\cos\phi}{\tan\theta}\right]$$

$$\boxed{\dot{\phi} = \omega_{z_o} - \frac{2\omega_t\sin\frac{\omega_n}{2}t\sin\left(\phi - \frac{\omega_n}{2}t\right)}{\tan\theta}} \tag{22}$$

Because of the initial conditions (18) we see that near zero:

$$\theta \rightarrow at^2 \qquad \text{and } \phi \rightarrow bt \tag{23}$$

Using these in (21) we have

$$at = 2\,\omega_t \frac{\omega_n}{2}\, t \qquad \text{or}$$

$$\dot{a} = \omega_t\, \omega_n \tag{24}$$

Then from (22):

$$b = \omega_{z_o} - \frac{2\omega_t\, \omega_n t\, (b - \frac{\omega_n}{2})\, t}{\omega_t\, \omega_n\, t^2} = \omega_{z_o} - 2\,(b - \frac{\omega_n}{2})$$

So

$$3b = (\omega_{z_o} + \omega_n)$$

So

$$b = \frac{1}{3}\,(\omega_{z_o} + \omega_n) \tag{25}$$

So near zero

$$\theta \rightarrow \omega_t\, \omega_n\, t^2$$

$$\phi \rightarrow \frac{1}{3}\,(\omega_{z_o} + \omega_n)\, t \tag{26}$$

Approximate Solution for A = B

Inspection of (22) shows that when $\omega_t << \omega_{z_o}$ and $\theta \neq 0$ that

$$\dot{\phi} \approx \omega_{z_o} \tag{27}$$

so that with the initial values (18):

$$\phi \approx \omega_{z_o} t \tag{28}$$

Using (28) in (21) we get

$$\dot{\theta} \approx 2\omega_t \sin \frac{\omega_n}{2} t \cos \left(\omega_{z_o} - \frac{\omega_n}{2}\right) t$$

$$= \omega_t \left[\sin\left[\frac{\omega_n}{2} + \left(\omega_{z_o} - \frac{\omega_n}{2}\right) \right] t - \sin\left[\left(\omega_{z_o} - \frac{\omega_n}{2}\right) - \frac{\omega_n}{2} \right] t \right]$$

$$\dot{\theta} \approx \omega_t \left[\sin (\omega_{z_o} t) - \sin (\omega_{z_o} - \omega_n) t \right] \tag{29}$$

Integrating with the initial values (18):

$$\boxed{\theta \approx - \frac{\omega_t \, \omega_n}{\omega_{z_o} (\omega_{z_o} - \omega_n)} - \frac{\omega_t}{\omega_{z_o}} \cos (\omega_{z_o} t) + \frac{\omega_t}{\omega_{z_o} - \omega_n} \cos (\omega_{z_o} - \omega_n) t} \tag{30}$$

Inspection of (3) shows that the maximum value of θ will be

$$\theta_{max} \approx - \frac{\omega_t \, \omega_n}{\omega_{z_o} (\omega_{z_o} - \omega_n)} - \frac{\omega_t}{\omega_{z_o}} - \frac{\omega_t}{\omega_{z_o} - \omega_n}$$

$$\boxed{\theta_{max} \approx - \frac{2\omega_t}{\omega_{z_o} - \omega_n}} \tag{31}$$

and this will occur when

$$\boxed{\cos \omega_{z_o} t_{max} = +1 \quad \text{and } \cos (\omega_{z_o} - \omega_n) t_{max} = -1} \tag{32}$$

Calculations

$$A = B = 110{,}000 \text{ slug ft}^2 \qquad \left\{ \text{CM/SM on return to earth} \right\}$$
$$C = 1/6\, A$$
$$M_y = 20{,}000 \text{ lb-ft} \qquad \left\{ \text{SM engine at } 20{,}000 \text{ lbs and } 1 \text{ ft. c. g. error} \right\} \tag{33}$$
$$\omega_{z_o} = \pi \text{ rad/sec} \qquad \left\{ \begin{array}{l} \text{Spin period} = 2 \text{ sec.} \\ \text{Centrifugal force at 3 ft} = r\omega_z^2 \\ = 1 \text{ earth g} \end{array} \right\}$$

Using (33) in (14):

$$\omega_n = \left(1 - \frac{1}{6}\right)\pi = 2.61/r/\text{sec}$$
$$\omega_t = \frac{20{,}000}{(110{,}000)(2.61)} = .07 \text{ rad/sec.} \tag{34}$$

Using these in (31)

$$\theta_{max} \approx \frac{(2)(.07)}{(.53)} = .265r = 15.1^\circ \tag{35}$$

at $\cos \pi\, t_{max} = +1$ and $\cos .53\, t_{max} = -1$

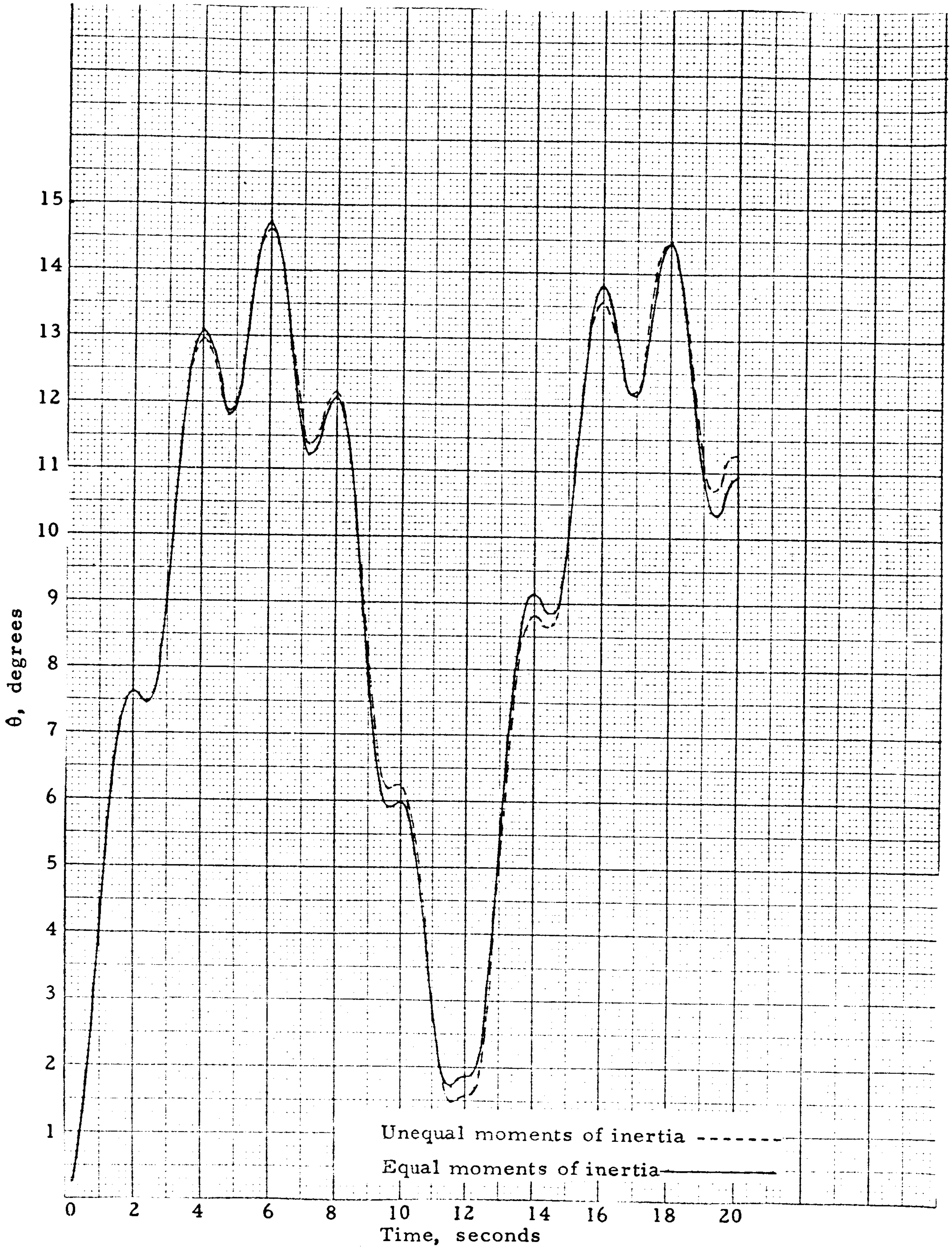
So

$$t_{max} \approx \frac{\pi}{.53}, \quad \frac{3\pi}{.53} = 6, \; 18 \text{ sec.} \tag{36}$$

with

$\pi\, t_{max} = 6\pi, \quad 18\pi$ cos cosine = + 1

The integration of the exact equations (21) and (22) for A = B and the values in (33) was carried out on a digital computer and the curve of θ versus time shown in Figure 2. From the curve we find max at 6 and 18



θ, degrees
15
14
13
12
11
10
9
8
7
6
5
4
3
2
1
0
2
4
6
8
10
12
14
16
18
20
Time, seconds
Unequal moments of inertia
Equal moments of inertia

Figure 2a

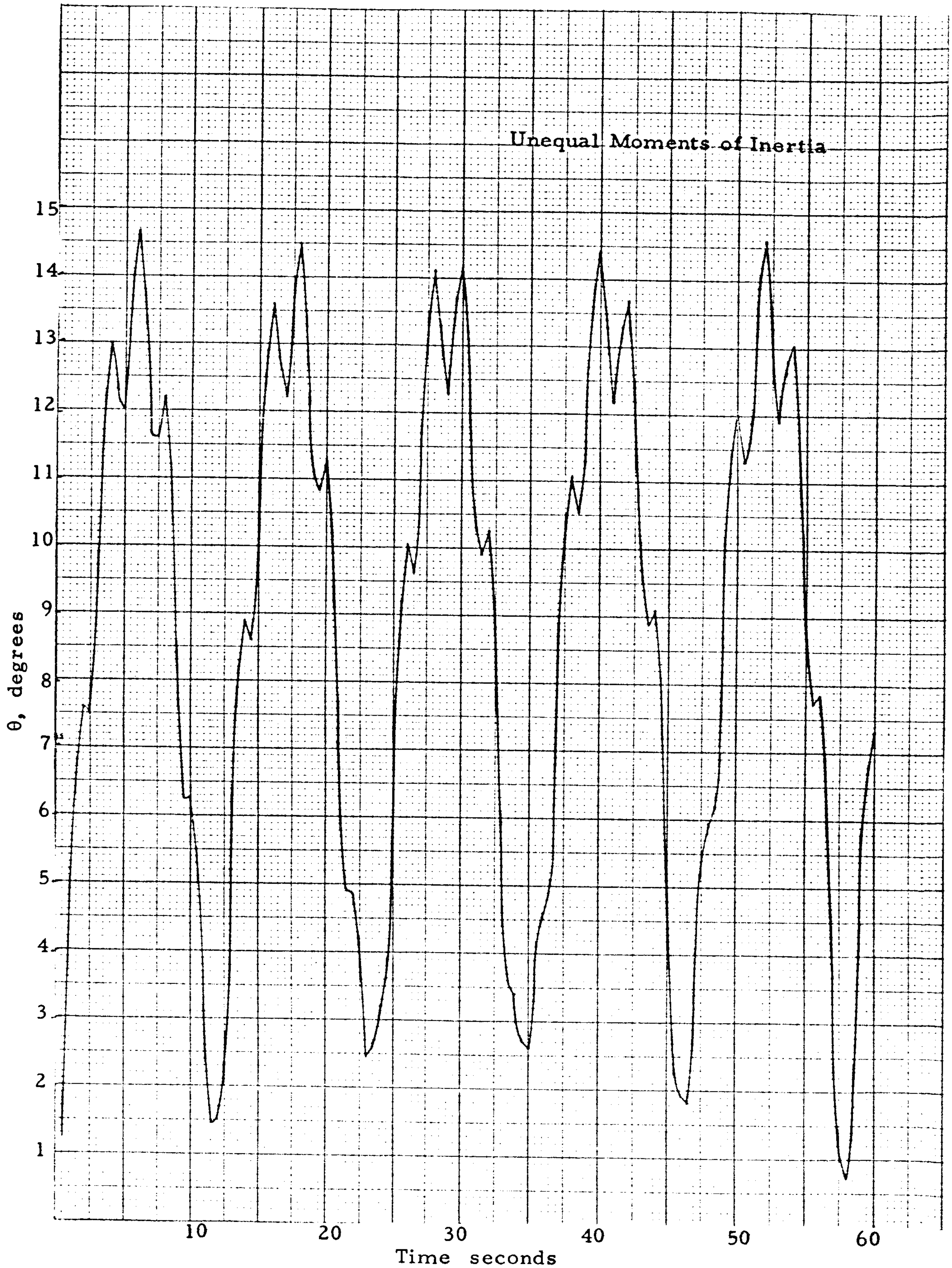

Unequal Moments of Inertia
15
14
13
12
11
10
9
8
7
6
5
4
3
2
1
θ, degrees
10
20
30
40
50
60
Time seconds

Figure 2b

sec with values of .257 and .252 radians thus agreeing surprisingly well with the approximate values.

Computer Results for $A \neq B$

To investigate the case of $A \neq B$ we let

$$\boxed{\begin{aligned} A &= (1.05)(110,000) = 115,500 \text{ slug ft}^2 \\ B &= (.95)(110,000) = 104,500 \text{ slug ft}^2 \end{aligned}}$$

so that $\dfrac{A - B}{\frac{1}{2}(A + B)} = \dfrac{11,000}{110,000} = 10\%$

with C, M_y and ω_{z_o} as before in (33).

The results of this calculation are shown superimposed on Figure 2. From the figure it is clear that the non-equality of A and B does not affect the motion in any important way.

Propellant Required to Spin up to ω_{z_o}

In the roll axis we have,

$$\dot{\omega}_z = \frac{F\ell}{C} \tag{37}$$

So

$$\omega_z = \frac{F\ell t}{C} = \frac{\ell}{C}\ (\text{Impulse}) \tag{38}$$

and

$$\omega + \text{Propellant} = \frac{\text{Impulse}}{I_{sp}} = \frac{C\,\omega_z}{\ell\, I_{sp}} \tag{39}$$

Letting $C = \frac{1}{6}\ (110{,}000) = 18{,}300$ slug ft^2

$$\ell = 6.5'$$

$$\omega_z = \pi \text{ rad/sec} \tag{40}$$

$$I_{sp} = 300 \text{ lb-sec/lb.}$$

$$\omega + \text{propellant} = \frac{(18{,}300)(\pi)}{(6.5)\ (300)} = 30 \text{ lbs.} \tag{41}$$

and another 30 lbs will be required to stop the spin after the thrusting period. Thus each correction maneuver will require 60 lbs of reaction jet propellant to establish and then remove the stabilizing spin.

Motion of $\overline{z}$ with Respect to $\overline{x}_o \cdot \overline{y}_o$

The preceding analysis gives the angle between $\overline{z}$ and $\overline{z}_o$ but does not tell how the thrust axis moves about in the plane normal to $\overline{z}_o$. To get this motion we could use the usual Euler angles but the equations are not well behaved. Instead, for computational purposes it turns out to be easier to define more general sets of direction cosines as

$$\begin{array}{lll} C_{11} = \overline{x} \cdot \overline{x}_o & C_{12} = \overline{x} \cdot \overline{y}_o & C_{13} = \overline{x} \cdot \overline{z}_o \\ C_{21} = \overline{y} \cdot \overline{x}_o & C_{22} = \overline{y} \cdot \overline{y}_o & C_{23} = \overline{y} \cdot \overline{z}_o \\ C_{31} = \overline{z} \cdot \overline{x}_o & C_{32} = \overline{z} \cdot \overline{z}_o & C_{33} = \overline{z} \cdot \overline{z}_o \end{array} \tag{42}$$

where the C_{13}, C_{23}, C_{33} set is the same as the set given by (8). There are 6 constant equations between these direction cosines:

$$
\begin{aligned}
C_{11}^2 + C_{12}^2 + C_{13}^2 &= x_o^2 = 1 \\
C_{12}^2 + C_{22}^2 + C_{32}^2 &= y_o^2 = 1 \\
C_{13}^2 + C_{23}^2 + C_{33}^2 &= z_o^2 = 1 \\
C_{11}^2 + C_{12}^2 + C_{13}^2 &= x^2 = 1 \\
C_{12}^2 + C_{22}^2 + C_{23}^2 &= y^2 = 1 \\
C_{13}^2 + C_{32}^2 + C_{33}^2 &= z^2 = 1
\end{aligned}
\tag{43}
$$

And these are useful for computational checks and to determine one from the other. With reference to Figure 3 and (42), we see that

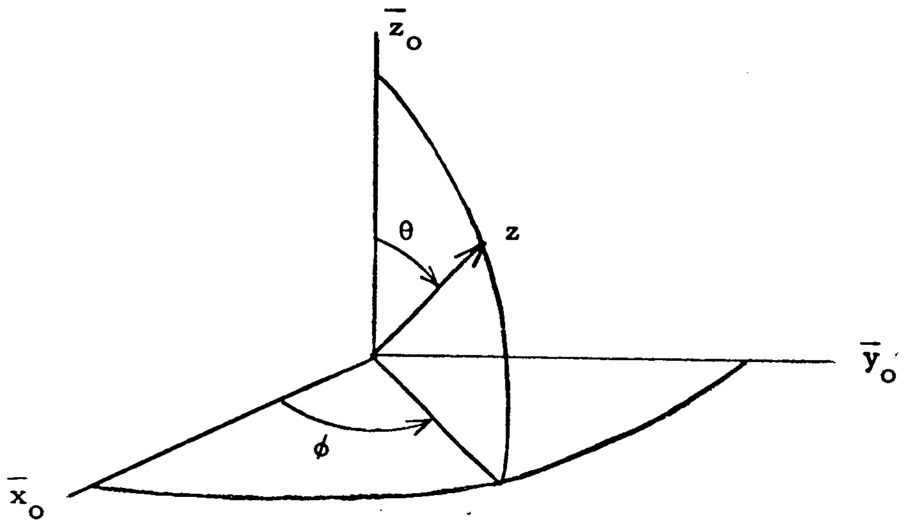


Figure 3.

$$
\boxed{
\begin{aligned}
\bar{z} \cdot \bar{x}_o &= C_{31} = \sin\theta \sin\phi \\
\bar{z} \cdot \bar{y}_o &= C_{32} = \sin\theta \cos\phi
\end{aligned}
}
\tag{44}
$$

So with C_{13} and C_{32} we know all about the motion of the $\overline{z}$ (thrust axis) with respect to the fixed $\overline{x}_o$, $\overline{y}_o$, $\overline{z}_o$ system. Differentiating (42), using the relation $\dot{\overline{x}} = \overline{\omega} \times \overline{x}$, and so forth, we get the sets:

$$\boxed{\begin{array}{ll} \dot{C}_{11} = C_{21}\omega_z - C_{31}\omega_y & \dot{C}_{12} = C_{22}\omega_z - C_{32}\omega_y \\ \dot{C}_{21} = C_{31}\,\omega_x - C_{11}\,\omega_z & \dot{C}_{22} = C_{32}\,\omega_x - C_{12}\,\omega_z \\ \dot{C}_{31} = C_{11}\,\omega_y - C_{21}\,\omega_x & \dot{C}_{32} = C_{12}\,\omega_y - C_{22}\,\omega_x \end{array}} \qquad (45)$$

With the initial conditions

$$\boxed{\begin{array}{l} C_{11}(o) = C_{22}(o) = 1 \\ C_{21}(o) = C_{31}(o) = C_{12}(o) = C_{32}(o) = 0 \end{array}} \qquad (46)$$

Now since the C's are direction cosines and we want to get C_{31} and C_{32}, we can transform the equations by the definitions below.

$$\begin{array}{ll} C_{31} \triangleq \sin\lambda_1 & C_{32} = \sin\lambda_2 \\ C_{11} \triangleq \cos\lambda_1 \cos\psi_1 & C_{22} = \cos\lambda_2 \cos\psi_2 \\ C_{21} = \cos\lambda_1 \sin\psi_1 & C_{12} = \cos\lambda_2 \sin\psi_2 \end{array} \qquad (47)$$

Hence from the initial conditions (46) we have

$$\boxed{\lambda_{10} = \lambda_{20} = \psi_{10} = \psi_{20} = 0} \qquad (48)$$

Using (47) in (45) we get

$$\begin{array}{l} \dot{C}_{31} = \dot{\lambda}_1 \cos\lambda_1 = \omega_y \cos\lambda_1 \cos\psi_1 - \omega_x \cos\lambda_1 \sin\psi_1 \\ \dot{C}_{32} = \dot{\lambda}_2 \cos\lambda_2 = \omega_y \cos\lambda_2 \sin\psi_2 - \omega_x \cos\lambda_2 \cos\psi_2 \end{array} \qquad (49)$$

So

$$\boxed{\begin{aligned} \dot{\lambda}_1 &= \omega_y \cos \psi_1 - \omega_x \sin \psi_1 \\ \dot{\lambda}_2 &= \omega_y \sin \psi_2 - \omega_x \cos \psi_2 \end{aligned}} \tag{50}$$

Solving for (40), (41) and differentiating using (45) and (47):

$$\dot{\psi}_1 = \frac{d}{dt} \tan^{-1} \left[\frac{C_{21}}{C_{11}} \right] = \frac{C_{11} \dot{C}_{21} - C_{21} \dot{C}_{11}}{C_{11}^2 + C_{21}^2}$$

$$= \frac{C_{11}(C_{31}\, \omega_x - C_{11}\, \omega_z) - C_{21}(C_{21}\, \omega_z - C_{31}\, \omega_y)}{C_{11}^2 + C_{21}^2}$$

$$\dot{\psi}_1 = - \omega_z + \frac{\omega_x \cos \lambda_1 \cos \psi_1 \sin \lambda_1 + \omega_y \cos \lambda_1 \sin \psi_1 \sin \lambda_1}{\cos^2 \lambda_1}$$

$$\boxed{\dot{\psi}_1 = - \omega_z + \tan \lambda_1 \left[\omega_x \cos \psi_1 + \omega_y \sin \psi_1 \right]} \tag{51a}$$

$$\dot{\psi}_2 = \frac{d}{dt} \tan^{-1} \left[\frac{C_{12}}{C_{22}} \right] = \frac{C_{22} \dot{C}_{12} - C_{12} \dot{C}_{22}}{C_{12}^2 + C_{22}^2}$$

$$= \frac{C_{22}(C_{22}\, \omega_z - C_{32} \omega_y) - C_{12}(C_{32}\, \omega_x - C_{12}\, \omega_z)}{C_{12}^2 + C_{22}^2}$$

$$= \omega_z - \left(\frac{\omega_x \cos \lambda_2 \sin \psi_2 \sin \lambda_2 + \omega_y \cos \lambda_2 \cos \psi_2 \sin \lambda_2}{\cos^2 \lambda_2} \right)$$

$$\boxed{\dot{\psi}_2 = \omega_z - \tan \lambda_2 \left(\omega_x \sin \psi_2 + \omega_y \cos \psi_2 \right)} \tag{51b}$$

Equation (50) and (51) together with the initial conditions then form a set of four first order equations which are well behaved and equally suitable for digital computation.

Approximate Solution for A = B Case

From Figure 3 we see that in any case of practical interest θ will have to be small, so that both λ_1 and λ_2 must be small and in particular will be small compared with ω_z. Thus in solving (51) it is always a good approximation to use just the leading term. In particular, for the first case considered, using (16) and neglecting the second terms in (51a) and (51b)

$$\dot{\psi}_1 \approx -\omega_{z_o} \quad \text{so} \quad \psi_1 = -\omega_{z_o} t$$
$$\dot{\psi}_2 \approx +\omega_{z_o} \quad \text{so} \quad \psi_2 = \omega_{z_o} t \tag{52}$$

Then using (52) and (16) in (50):

$$\dot{\lambda}_1 \approx \omega_t \sin \omega_n t \cos \omega_{z_o} t + \omega_t (1 - \cos \omega_n t) \sin \omega_{z_o} t$$
$$= \omega_t \sin \omega_{z_o} t - \omega_t \sin (\omega_{z_o} - \omega_n) t$$
$$\dot{\lambda}_2 \approx \omega_t \sin \omega_n t \sin \omega_{z_o} - \omega_t (1 - \cos \omega_n t) \cos \omega_{z_o} t \tag{53}$$
$$\approx -\omega_t \cos \omega_{z_o} t + \omega_t \cos (\omega_{z_o} - \omega_n) t$$

Integrating (53) with the initial conditions (48):

$$\boxed{\begin{aligned} \lambda_1 &\approx \frac{\omega_t}{\omega_{z_o}} (1 - \cos \omega_{z_o} t) - \frac{\omega_t}{\omega_{z_o} - \omega_n} (1 - \cos (\omega_{z_o} - \omega_n) t) \\ \lambda_2 &\approx -\frac{\omega_t}{\omega_{z_o}} (\sin \omega_{z_o} t) + \frac{\omega_t}{\omega_{z_o} - \omega_n} \sin (\omega_{z_o} - \omega_n) t \end{aligned}} \tag{54}$$

Thus

$$\boxed{\begin{aligned} \lambda_{1_{avg}} &= -\frac{\omega_t \omega_n}{\omega_{z_o} (\omega_{z_o} - \omega_n)} \\ \lambda_{2_{avg}} &= 0 \end{aligned}} \tag{55}$$

From the form of (54) we see that if we look down on the (x_o, y_o, z_o) system and trace the motion of the z axis it has a constant average value along the $-\bar{x}_o$ axis and is the sum of two sinusoids. The high frequency ω_{z_o} term is of small amplitude and the low frequency $(\omega_{z_o} - \omega_n)$ term is of larger amplitude. Using the values of (34)

$$\frac{\omega_t \omega_n}{\omega_{z_o}(\omega_{z_o} - \omega_n)} = \frac{(.07)(2.61)}{3.14(.53)} = .11\ r = 6.3^\circ$$

$$\frac{\omega_t}{\omega_{z_o}} = \frac{.07}{3.14} = .022\ r = 1.3^\circ$$

$$\frac{\omega_t}{\omega_{z_o} - \omega_n} = \frac{.07}{.53} = .132 = 7.6^\circ \tag{56}$$

$$\omega_{z_o} = 3.14\ r/sec = 2\ sec\ period$$

$$\omega_{z_o} - \omega_n = 0.53\ r/sec = 12\ sec\ period$$

Thus we can construct the motion of the z axis as in Figure 4.

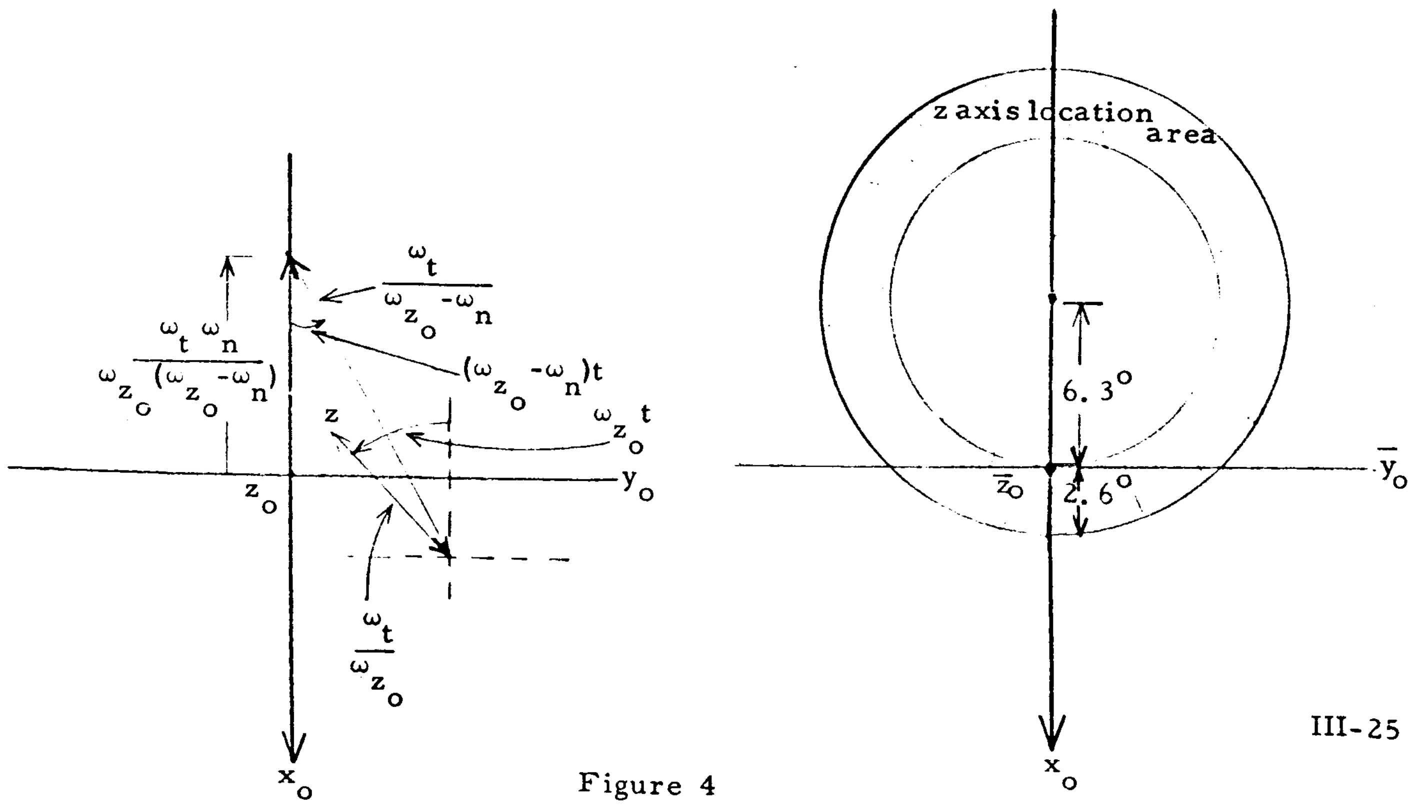


Figure 4

To see the effect of spin speed, we re-express the average angle in terms of torques, etc., as

$$\lambda_{1_{avg}} = \frac{\omega_t \omega_n}{\omega_{z_o}(\omega_{z_o} - \omega_n)} = \frac{\frac{M}{A\omega_n} \cdot \omega_n}{\omega_{z_o}(\omega_{z_o} - \omega_{z_o}(1 - \frac{C}{A}))} = \frac{M}{A\,\omega_{z_o}^2 (\frac{C}{A})}$$

$$\boxed{\lambda_{1_{avg}} = \frac{M}{C\,\omega_{z_o}^2}} \qquad (57)$$

Thus for a given moment the average z axis displacement from the desired direction varies inversely as the square of the spin speed.

We now recall the results of Note 22 where it was shown that when we are only controlling re-entry angle then we can place $\overline{z_o}$ along the re-entry angle sensitive axis as defined by Note 22, and then the motion of $\overline{z}$ in the $\overline{x_o}$, $\overline{y_o}$ plane contributes zero error to re-entry angle so all we are concerned with is $(1 - \overline{z} \cdot \overline{z_o})$ and this is less than a few percent even for the 12'' c.g. error considered.

Then, while more analysis is needed, this preliminary study strongly suggests that for CM aborts, spin stabilization for attitude control during the boost phases is very promising.

Incidentally, in this regard we note that the main engine maximum gimbal deflection is $\pm 7^o$ and the distance from the nozzle to the c.g. of the CM/SM is about 15 ft. Therefore, with the engine jammed hard over, the lever arm of main engine thrust with respect to the vehicle center line is $15 \sin 7^o = 1.8$ ft. Thus, with a c.g. shift of 1 ft. the maximum possible lever arm with jammed engine is 2.8 ft. rather than the 1 ft. assumed in preceding analysis. If we then raise the spin speed to $\sqrt{2.8}\ \pi = 5.25$ rad/sec, so the centrifugal force 3 ft. off center is up to 2.8 g, the z axis deflections will still be the same as in the analysis. Thus with spin stabilization it appears possible to operate with a hard over jammed main engine.

APOLLO NOTE NO. 47

G. F. Floyd
April 17, 1963

EFFICIENT BOOSTING WITH LOW THRUSTS

This note examines the feasibility of CM/SM return to earth from lunar orbit in the event of a main engine propulsion failure. In this case, the only propulsive devices available are the reaction jets, and of the 16 reaction jets only 4 are oriented so as to apply positive thrust. Since their individual levels are 100# the thrust available is 100, 200, 300, or 400# depending upon how many of the possible 4 are still working. The reaction jets are designed to be continuously burned without overheating and use the same fuel as the main engine. However, in the present design plumbing is not provided to replenish the reaction jet tanks from the main engine fuel tanks. However, this provision was in the original design and perhaps could be provided.

The CM/SM weight in lunar orbit is about 44,500 # and after the 3000 ft/sec boost needed for transearth trajectory will weigh about 36,500#. Thus the average mass is about 1200 slugs, so even with all the possible thrusting reaction jets the average boost acceleration is only about 1/3 ft/sec^2 so to gain 3000 ft/sec will take at least 9000 sec or 2.5 hours which is more than an orbit period, and this could be increased to 10 hours or 5 orbit period if only one thrust producing reaction jet were still working. Thus we have a low thrust case and wish to investigate the boost required to achieve the proper transearth trajectory. Since we are only interested in calculating the extra boost needed, we consider the simple analytical case of going from circular to escape with finite thrust. For completeness in the notes we next derive a well known but useful relation,

Let

$$E \triangleq 1/2\, V^2 - \frac{\mu}{R} \tag{1}$$

$$= 1/2\, \overline{V} \cdot \overline{V} - \frac{\mu}{R}$$

Then

$$\dot{E} = \bar{V} \cdot \dot{\bar{V}} + \frac{\mu}{R^2} \dot{R} \tag{2}$$

But

$$\dot{\bar{V}} = \bar{a} + \bar{g} = \bar{a} - \frac{\mu}{R^2} \bar{R} \tag{3}$$

$$\dot{R} = \bar{V} \cdot \bar{1}_R = \frac{\bar{V} \cdot \bar{R}}{R} \tag{4}$$

so

$$\dot{E} = \bar{V} \cdot \left[\bar{a} - \frac{\mu}{R^2} \bar{R}\right] + \frac{\mu}{R^2} \frac{\bar{V} \cdot \bar{R}}{R} \tag{5}$$

$$= \bar{V} \cdot \bar{a}$$

Also

$$V_b = \int_o^t a \, dt \tag{6}$$

So

$$\dot{V}_b = a . \tag{7}$$

Hence dividing (5) by (7) we have

$$\frac{dE}{dV_b} = \frac{\dot{E}}{\dot{V}_b} = \frac{\bar{V} \cdot \bar{a}}{a} = \bar{V} \cdot \bar{1}_a \tag{8}$$

Then letting (δ) be the angle between the instantaneous velocity and the instantaneous thrust acceleration, we have

$$\bar{1}_V \cdot \bar{1}_a \triangleq \cos \delta \tag{9}$$

So solving (1) for V and using this and (9) in (8) we get

$$\frac{dE}{dV_b} = V \cos \delta = \sqrt{\frac{2\mu}{R} + 2E} \quad \cos \delta \tag{10}$$

So

$$d V_b = \frac{\sec \delta (d E)}{\sqrt{\frac{2\mu}{R} + 2E}} \tag{11}$$

Integrating (11) over the energy levels E_o to E_f we get the required boost V_b :

$$\boxed{V_{b\ req.} = \int_{E_o}^{E_f} \frac{\sec \delta \, dE}{\sqrt{\frac{2\mu}{R} + 2E}}} \tag{12}$$

Equation (12) involves no approximations and is useful because it clearly shows what makes boosting slowly inefficient in that (δ) is under our control so can be made zero, and then the only controllable is (R) since E has to go from E_o to E_f. From (12) we see that impulsive boosting with $\delta = 0$ gives

$$V_{b\ imp.} = \int_{E_o}^{E_f} \frac{dE}{\sqrt{\frac{2\mu}{R_o} + 2E}} = \sqrt{\frac{2\mu}{R_o} + 2E} \Bigg]_{E_o}^{E_f} \tag{13}$$

Then for circular to escape we have

$$\begin{aligned} E_o &= E_c = -\frac{\mu}{2R_o} \\ E_f &= E_e = 0 \end{aligned} \tag{14}$$

so

$$V_{b\ imp.} = \sqrt{\frac{2\mu}{R_o}} - \sqrt{\frac{\mu}{R_o}} = \sqrt{\frac{\mu}{R_o}} \left(\sqrt{2} - 1\right) \tag{15}$$

The procedure to be followed is suggested by (12) and is to hold $\bar{a}$ slightly below $\dot{\bar{V}}$, $(\delta = \delta_c)$, to try and hold $\bar{R}$ down without increasing sec δ significantly. We then pick a value R_c greater than R_o and adopt the following rule:

$$\boxed{\text{Boost at constant down } (\delta_c) \text{ when } R \leq R_c} \tag{16}$$

so when R reaches R_c we cut off and coast until R becomes less than R_c.

Before examining the type of trajectory that will result we return to (12) to see what kinds of values of R_c and δ_c we might want. Since the rule (16) will result in R always being less than R_c while we are boosting we have:

$$V_{b\,req.} < \int_{E_o}^{E_f} \frac{\sec \delta_c \, dE}{\sqrt{\frac{2\mu}{R_c} + 2E}} = \sec \delta_c \sqrt{\frac{2\mu}{R_c} + 2E}\,\Bigg]_{-\frac{\mu}{2R_o}}^{o}$$

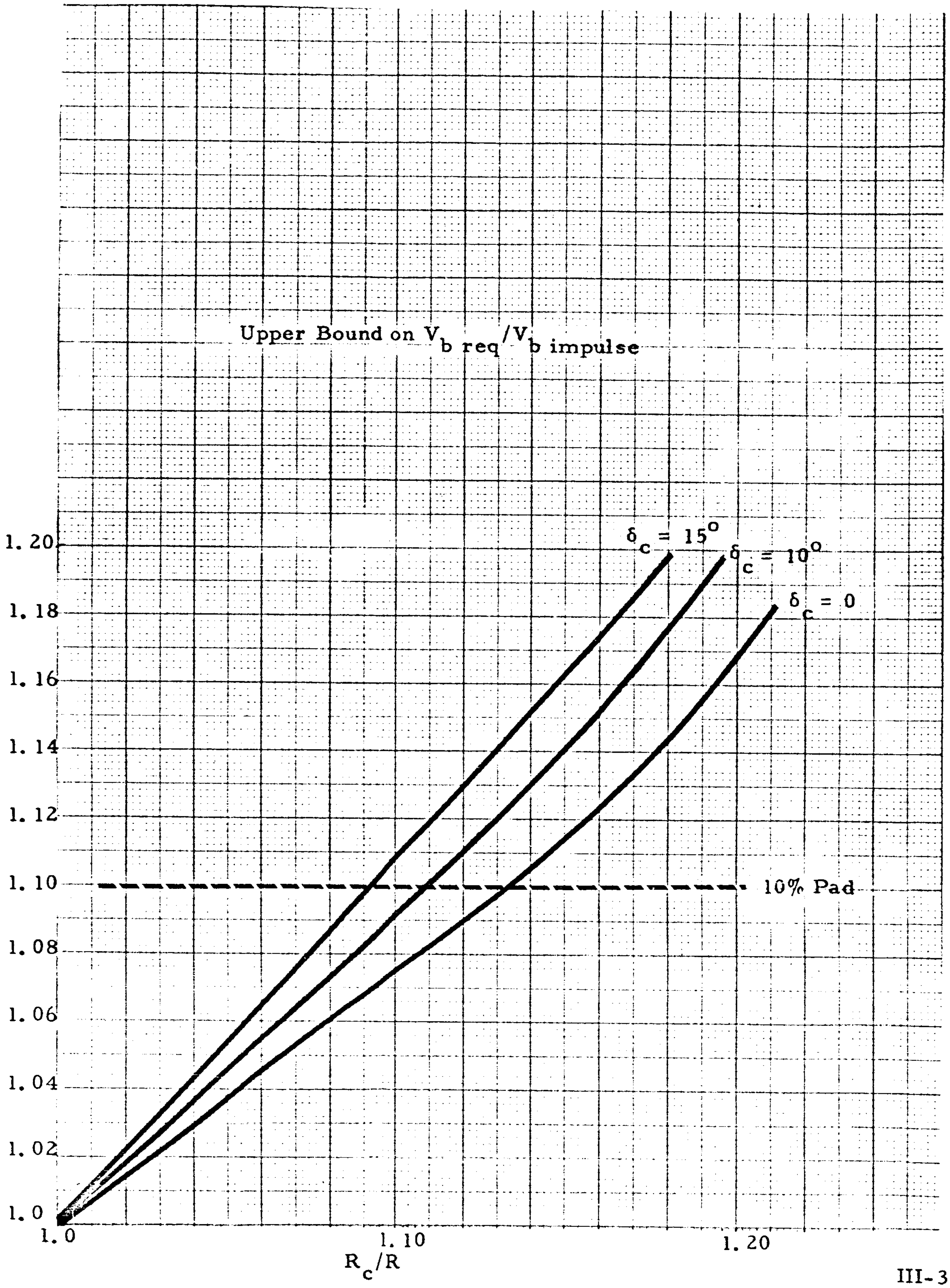
$$= \sec \delta_c \left(\sqrt{\frac{2\mu}{R_c}} - \sqrt{\frac{2\mu}{R_c} - \frac{\mu}{R_o}} \right) \tag{17}$$

$$= \sqrt{\frac{\mu}{R_c}} \sec \delta_c \left[\sqrt{2} - \sqrt{2 - \frac{R_c}{R_o}} \right]$$

Taking the ratio of (17) to (15) we get

$$\boxed{\frac{V_{b\,req.}}{V_{b\,imp.}} < \sqrt{\frac{R_c}{R_o}} \sec \delta_c \left(\frac{\sqrt{2} - \sqrt{2 - \frac{R_c}{R_o}}}{\sqrt{2} - 1} \right)} \tag{18}$$

Equation (18) is plotted in Figure 1 and from the figure we see that to hold the upper band or $V_{b\,req.}$ within 10% of $V_{b\,imp.}$ requires that we keep δ_c less than about 15° and R_c less than about 1.10 R_o. For the lunar case



Upper Bound on $V_{b\,req}/V_{b\,impulse}$
$\delta_c = 15^o$
$\delta_c = 10^o$
$\delta_c = 0$
10% Pad
1.20
1.18
1.16
1.14
1.12
1.10
1.08
1.06
1.04
1.02
1.0
1.0
1.10
1.20
R_c/R

Figure 1.

R_o is about 1000 miles so that the boosts will be terminated after about a 100 mile increase in altitude above the circular orbit altitude. Thus for a 100 mile orbit altitude the boosts will be made at altitudes between 100 and 200 miles.

To investigate the boost trajectory that results we refer to Figure 2.

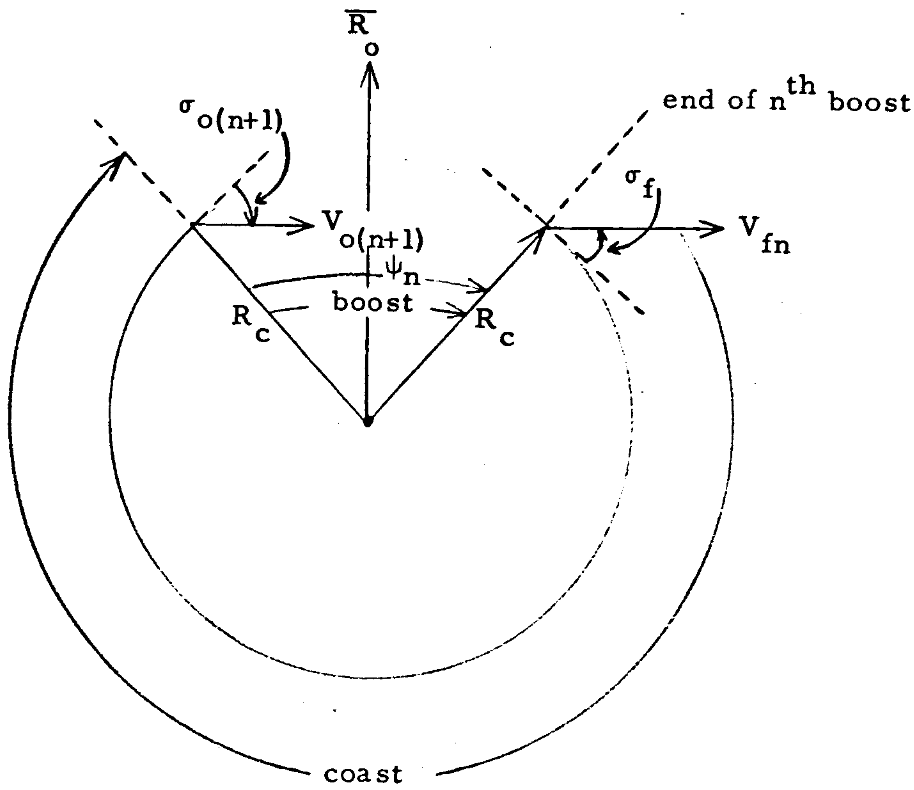


Figure 2.

During the coast phase following the n^{th} boost cycle, energy and momentum are constant so

$$E_{o\,(n+1)} = E_{f\,(n)}$$

or

$$1/2\, V^2_{o\,(n+1)} - \frac{\mu}{R_c} = 1/2\, V^2_{fn} - \frac{\mu}{R_c} \, .$$

Hence

$$V_{o(n+1)} = V_{fn} \tag{19}$$

Also

$$H_{o(n+1)} = H_{f(n)}$$

$$R_c \, V_{o(n+1)} \cos \sigma_{o(n+1)} = R_c \, V_{fn} \cos \sigma_{fn}$$

so with (19)

$$\sigma_{o(n+1)} = \sigma_{fn}$$

with the signs of σ_o and σ_f taken oppositely as in the figure. Thus with this procedure the coast phase simply changes the sign of pitch angle to a dive angle of the same magnitude. Therefore, on a computer when we integrate the equations of motion, we simply forget the coast phase and integrate the ordinary equations of motion, changing the sign of $\dot{R}$ every time R reaches R_c.

To investigate the motion of the boost portion around the moon itself we use as an inertial reference $\overline{R}_o$. Then let ψ_1 be the central angle that the vehicle moves through in the first boost. Assuming approximate symmetry about $\overline{R}_o$ of the elliptical coast phases we then see that the second boost starts approximately at ψ_1 before $\overline{R}_o$ so the vehicle ends the second boost at $(\psi_2 - \psi_1)$ beyond $\overline{R}_o$, so starts the third at $(\psi_2 - \psi_1)$ before $\overline{R}_o$, so ends at $(\psi_3 - \psi_2 + \psi_1)$ beyond. Thus the angular distance beyond $\overline{R}_o$ that the boosts end follow the pattern ψ_1, $\psi_2 - \psi_1$, $\psi_3 - \psi_2 + \psi_1$, $\psi_4 - \psi_3 + \psi_2 - \psi_1$, etc. Since (ψ_n) is steadily decreasing, the algebraic sum of the ψ_n also goes to zero with this method. Thus the boost tends to end at $\overline{R}_o$, just as an impulsive boost would.

Having shown that the actual boost tends to take place at a fixed point with respect to the moon, it is now convenient to piece together all the boost phases. This leads to the type of figure sketched in Figure 3.

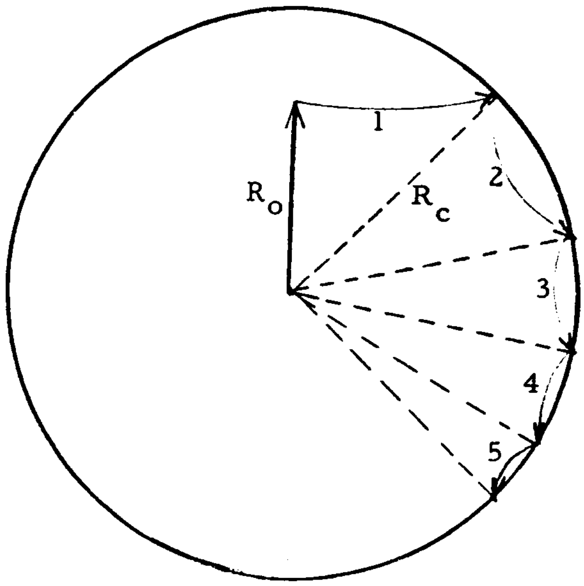


Figure 3.

The minimum R during a phase can of course be less than R_o depending on the value of δ_c that is used. To investigate this a little analytically we use the definition of symbols of Figure 4.

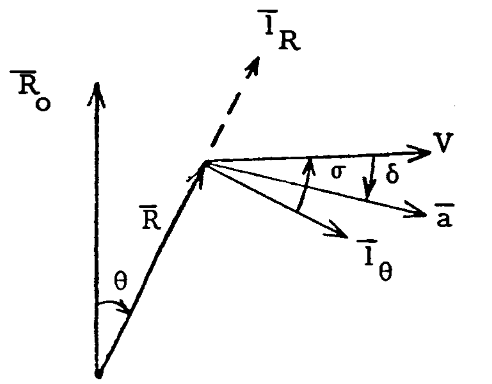


Figure 4.

Then we have the usual equations

$$\ddot{\bar{R}} = \bar{a} + \bar{g} \tag{20}$$

where

$$\dot{\bar{R}} = \bar{I}_R \dot{R} + \bar{I}_\theta R \dot{\theta} \tag{21}$$

$$\bar{a} = \bar{I}_\theta a \cos(\sigma - \delta) + \bar{I}_r a \sin(\sigma - \delta) \tag{22}$$

$$\bar{g} = -\bar{I}_r \mu/R^2 \tag{23}$$

$$\bar{I}_\theta = \bar{I}_r \dot{\theta} \;\; ; \;\; \dot{\bar{I}}_r = -\bar{I}_\theta \dot{\theta} \tag{24}$$

Differentiating (21), then using (20), (22), and (23) we get the usual equation for $\ddot{R}$

$$\ddot{R} = R\dot{\theta}^2 - \mu/R^2 + a \sin(\sigma - \delta) \tag{25}$$

Where from the figure

$$R\dot{\theta} = V \cos\sigma \tag{26}$$

Using this in (25) we have

$$R = \frac{V^2 \cos^2\sigma}{R} - \frac{\mu}{R^2} + a \sin(\sigma - \delta) \tag{27}$$

Now at the start:

$$\frac{V_o^{\,2}}{R_o} = \frac{\mu}{R_o^{\,2}} = 1 \text{ moon g} \approx 5.5 \text{ ft/sec}^2$$

and near the end $V \rightarrow \sqrt{2}\, V_o$ with $R \approx R_c$, so

$$\frac{V^2}{R} \longrightarrow 2 \text{ moon g} \approx 11 \text{ ft/sec}^2$$

with μ/R^2 staying at about 5.5 ft/sec^2. For the case of interest (a) was less than .3 ft/sec^2, and we now know from the first analysis that we want less than 15^o or so. Thus, during all except possibly the first cycle, the last term in (27) is small compared to the sum of the first two and this suggests that $\ddot{R}$, hence the time duration of each cycle, cannot be controlled appreciably with δ. So $\delta = 0$ should be about as good as anything else.

A digital computer computation of the motion has been made for several cases and the results presented in Figure 5 for the following conditions:

R_o = Lunar Circular Orbit at 100 n. mi. altitude.

ω_o = 44, 500 lbs.

I_{sp} = 300 lb-sec/lb.

Thrust = 400 lbs and 20, 000 lbs.

$\delta = 0^o$

From the figure we see that the procedure is actually much more efficient than the upper bound plotted in Figure 1, however, the total time required is very long. The boost time is, of course, still about 9000 sec but the coast periods are very long as the orbit eccentricities approach one. The number of orbits is reasonable (the spacecraft reaches escape during its third orbit for $R_c - R_o$ = 500 n. mi.), and these orbits remain in the lunar sphere of influence so the calculations are correct.

The total time to reach escape is, however, probably excessive in terms of the life-support system capability except for the case where the CM main engine fails during the terminal portion of the deboost <u>into</u> lunar orbit. Here an abort will be necessary and we have a 24 hour head start so the method may be useful.

It is quite apparent from the nature of these first results that very good efficiency and short boost times would result with thrust levels down by an order of magnitude below the present 20, 000 lbs. To investigate this, Figure 6 shows fuel cost versus thrust level for continuous boosting from circular to escape at $\delta = 0^o$. Also shown are the fuel costs to go continuously from circular to higher eccentricities. From the figure it is clear that the extra fuel cost of lower thrusts is small and can be more than made up with the reduced engine weight which would accompany lower thrusts.

Circular to Escape

400 lb Thrust at $\delta = 0^{o}$

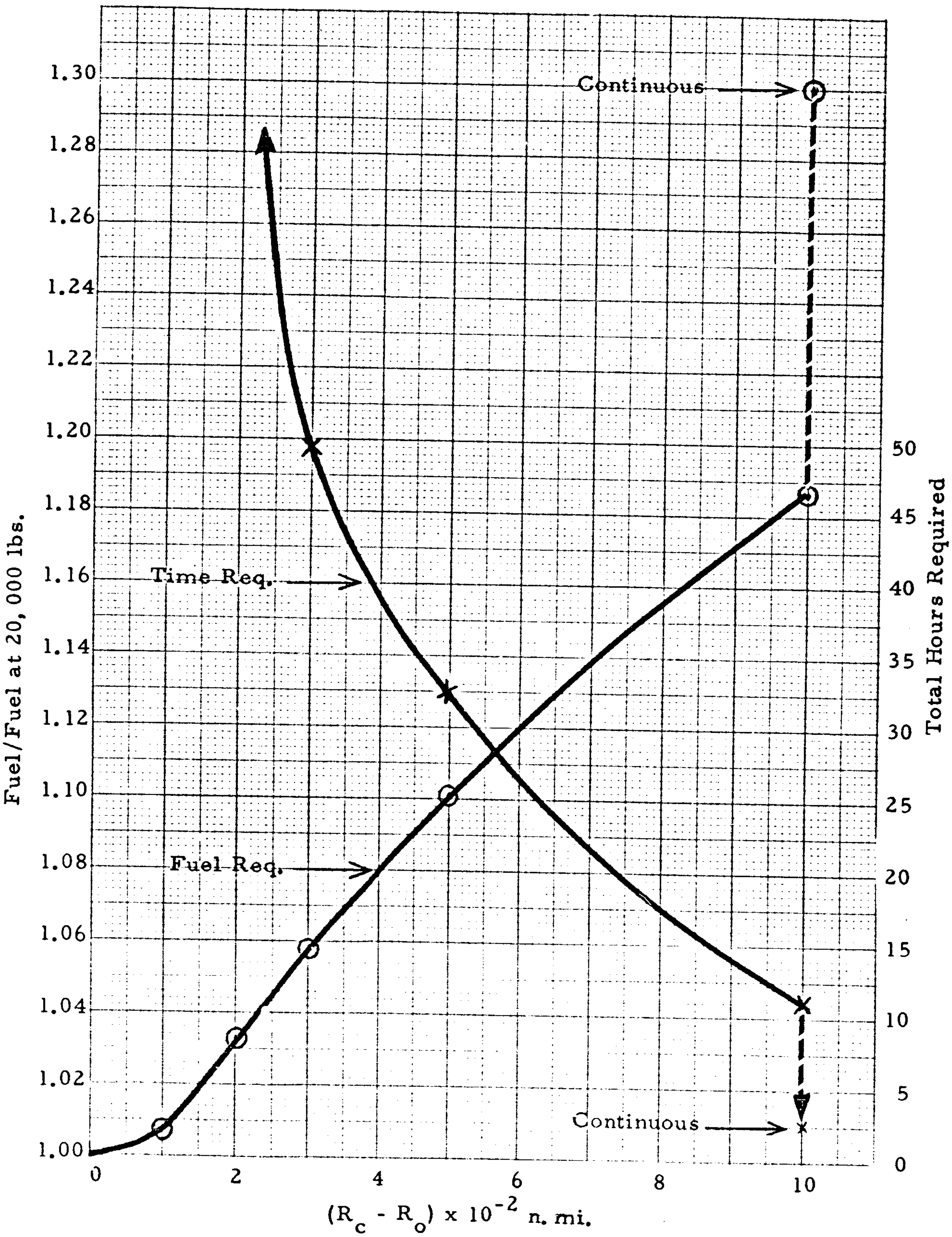
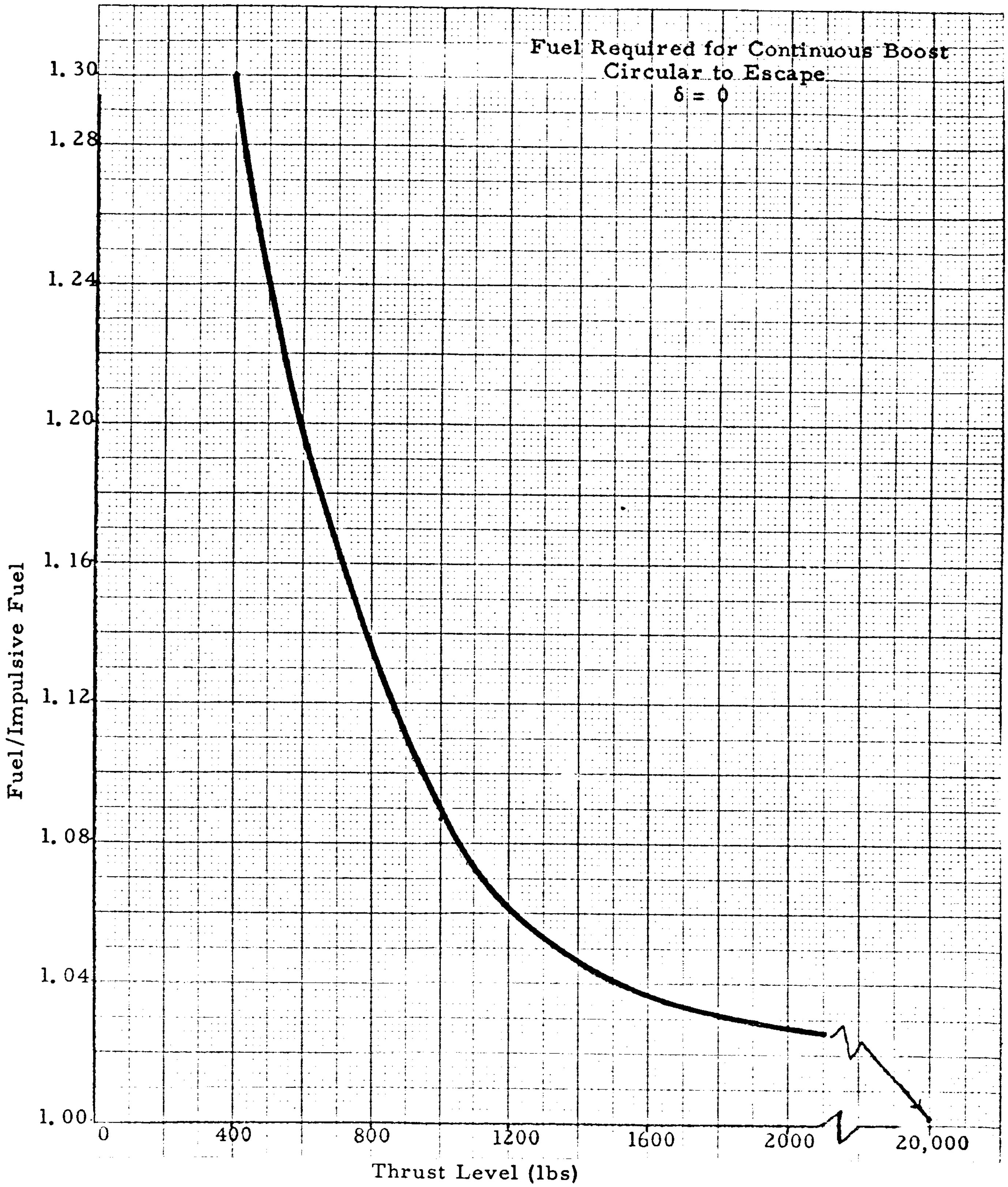


Figure 5.



Fuel Required for Continuous Boost
Circular to Escape
$\delta = 0$
Fuel/Impulsive Fuel
1.30
1.28
1.24
1.20
1.16
1.12
1.08
1.04
1.00
0
400
800
1200
1600
2000
20,000
Thrust Level (lbs)

Figure 6.

More important, a thrust level of 2000 lbs instead of 20, 000 lbs would increase the accuracy and ease of the emergency mode attitude control by a factor of 10 and also permit small midcourse corrections to be made with the main engine.

APOLLO NOTE NO. 49

G. F. Floyd
23 April 1963

PRE-BOOST ATTITUDE CONTROL

In Note No. 38 we showed that spin stabilization appears quite feasible for attitude control during a boost phase, and that this form of stabilization should be capable of handling even the case of a hard-over jammed main engine. To complete the attitude control study, this note is devoted to an analysis of how the vehicle can be initially aligned in the commanded direction before being spun up and the main boost applied.

If the reaction jet and autopilot are operating there is, of course, no control problem but the purpose of this note is to examine how much of the attitude control system can be inoperative and still achieve the initial alignment. The reaction jet nozzles are mounted on the skin with the vehicle c. g. in the nozzle plane in four clusters of 4 nozzles each. Thus, we have 16 nozzles, and with this arrangement have the following force applicators (neglecting thrust components):

4 CW Roll
4 CCW Roll
2 Pitch Down
2 Pitch Up (1)
2 Yaw Right
2 Yaw Left

Since each jet has a separate combustion chamber and a separate pair of relay operated control valves to admit the fuel and oxidizer to its combustion chamber, the nozzle operating probabilities are essentially independent (neglecting common manifold failures). Thus, there is a definite probability that the reaction jet system will be in a degraded operating condition. Let (q) be the probability that any particular reaction jet nozzle is <u>not</u> in operating condition, and this will be taken as the same number for all 16 nozzles.

Then, the probability that we <u>have</u> attitude control in a particular direction is as follows:

Have CW Roll $= 1 - q^4$
Have CCW Roll $= 1 - q^4$
Have Pitch Down $= 1 - q^2$

Have Pitch Up $= 1 - q^2$

Have Yaw Right $= 1 - q^2$

Have Yaw Left $= 1 - q^2$

If we forget the failures of manifolds (so sets go out), the above are independent, hence the probability that we have complete control (all axes both plus and minus) is just the product, so

$$P_{complete} = (1 - q^4)^2 (1 - q^2)^4 \qquad (2)$$

If $q = 0.1$, $P_{complete}$ is only .96 which is nothing spectacular in the light of the 0.99 crew safety requirement.

It, therefore, appears of interest to see how many of the complete reaction jet control nozzles can be inoperative and still perform the necessary pre-boost attitude control. Calling the 8 pitch and yaw nozzles fore-aft nozzles, we will next show that starting from arbitrary initial body angles and body rates, it is possible to achieve the desired final orientation of the thrust with the use of:

1 CW Roll
1 CCW Roll (3)
1 Fore-aft Jet (can be either + or -)

Calling this set the necessary set, and letting (q) be as before the failure probability of any of the 16 nozzles, the probability that we have the necessary nozzles is

$$P_{necessary} = (1 - q^4)^2 (1 - q^8) \qquad (4)$$

Again, taking $q = 0.1$, $P_{necessary} = .9998$ which is a good number but when $q = .3$, $P_{necessary}$ falls to .984.

In an effort to improve the above situation, it will also be shown that the necessary attitude control can be achieved by the sequential use of the 3 directions called out in equation (3), and further that the allowable time delay between the use of each of these three can be minutes or longer. Therefore, if the individual manifold assemblies could be manually rotated through $\pm 180^\circ$, it would be possible to use any of the 16 jets for any of the three functions listed. Calling this arrangement the modified system, the probability of it working is

$$P_{modified} = 1 - q^{16} \tag{5}$$

To see whether this rather complicated engineering modification would really pay dividends we have Figure 1 which shows the three probabilites of equations (2), (4) and (5) versus (q).

Inspection of the figure shows that unless (q) is very small indeed, then it is unrealistic to expect the complete system to be operative with probabilities consistent with the <u>overall</u> mission crew safety requirement of 0.99. In fact, since the attitude control is necessary for a safe return, the probability gains to be made with the modified system are quite significant and an engineering study should be made of the difficulties involved.

It is now necessary to demonstrate that the necessary attitude control can be accomplished with the minimal system described above. For this problem we use the same definitions as in Note No. 38 and consider the case of the symmetrical vehicle (A = B) with the z axis the longitudinal spacecraft axis. We first consider the case where we apply roll moments only so $M_x = M_y = 0$ and M_z will not be assumed constant. The equations of motion are derived as in Note No. 38 and with the above torque assumptions become for (A = B):

$$\begin{aligned} M_x &= 0 = A\dot{\omega}_x - (A - C)\,\omega_z\,\omega_y \\ M_y &= 0 = A\dot{\omega}_y + (A - C)\,\omega_z\,\omega_x \\ M_z &= C\,\dot{\omega}_z \end{aligned} \tag{6}$$

So

$$\begin{aligned} \dot{\omega}_x &= (1 - \frac{C}{A})\,\omega_z\omega_y \\ \dot{\omega}_y &= -(1 - \frac{C}{A})\,\omega_z\omega_x \\ \dot{\omega}_z &= \frac{M_z}{C} \end{aligned} \tag{7}$$

Now use the definitions

Attitude Control Probability

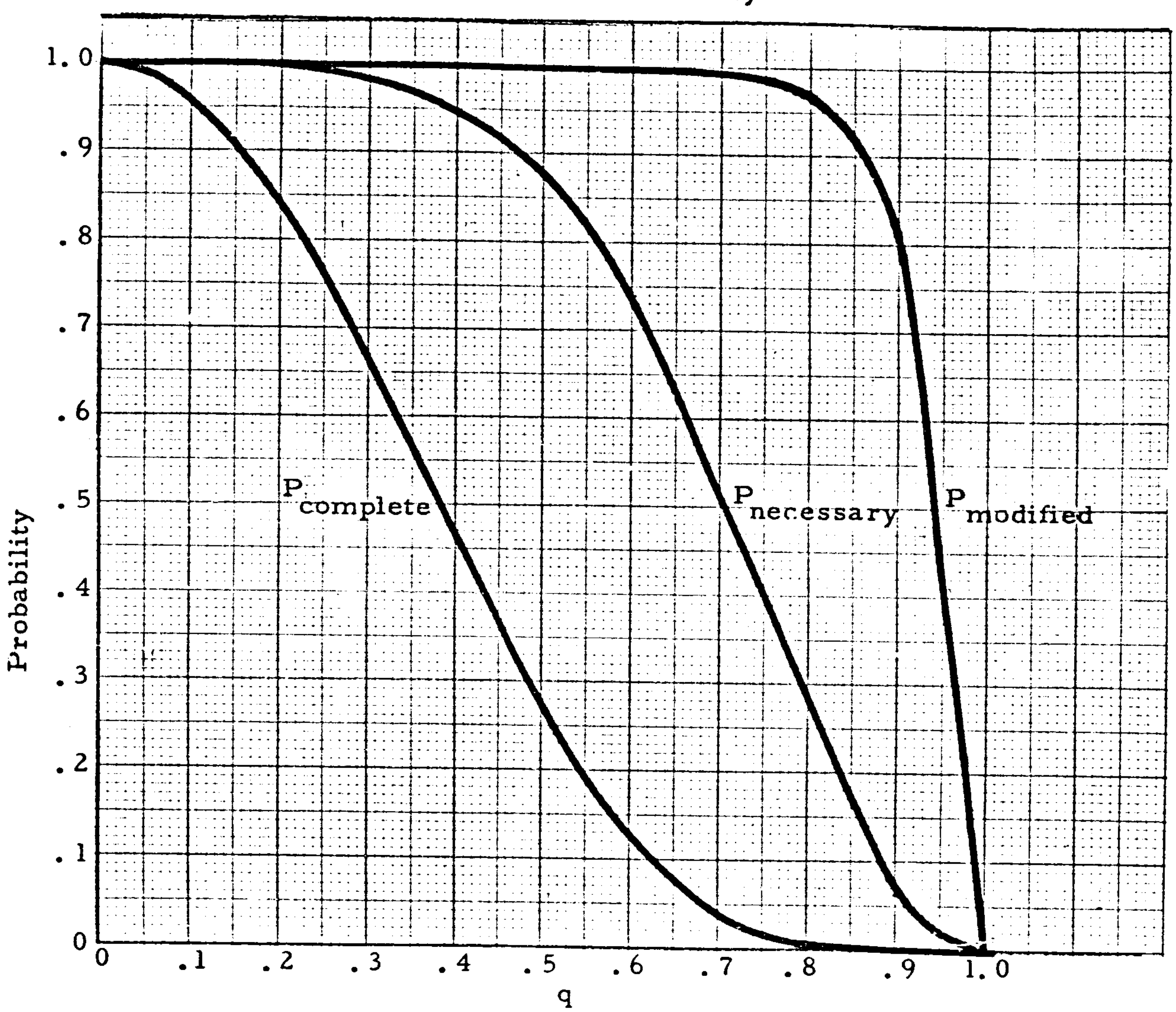


Figure 1.

$$\omega_{xy_o} \triangleq \sqrt{\omega_{x_o}^{2} + \omega_{y_o}^{2}} \tag{8}$$

$$\phi(t) \triangleq \tan^{-1}\left[\frac{\omega_{y_o}}{\omega_{x_o}}\right] - (1 - \frac{C}{A}) \int_o^t \omega_z \, dt \tag{9}$$

With these definitions the solution of (7) is

$$\boxed{\begin{aligned} \omega_x(t) &= \omega_{xy_o} \cos \phi(t) \\ \omega_y(t) &= \omega_{xy_o} \sin \phi(t) \\ \omega_z(t) &= \omega_{z_o} + \frac{1}{C} \int_o^t M_z \, dt \end{aligned}} \tag{10}$$

as may be easily checked by substitution into (7) and an initial condition check.

Now let ψ be the roll angle about z so,

$$\dot{\psi} \triangleq \omega_z \tag{11}$$

Then with (9) we have

$$\boxed{\Delta\phi = - (1 - \frac{C}{A}) \Delta \psi} \tag{12}$$

With these equations we now assume that either we have only one operable reaction jet which can be rotated into either the CW or CCW roll or a fore-aft direction or a fixed jet in each of these three directions. Let the initial conditions be an arbitrary spin of the spacecraft about all three axes and let there be an arbitrary direction in space along which we wish to point the $\overline{z}$ axis. The procedure is as follows:

Step 1 Apply roll torque until the spin about the $\bar{z}$ axis reaches zero.

The angular velocity will now be in the xy plane (since ω_z is now zero) and $(\dot{\phi})$ will also be zero since $\omega_z = 0$. (See (9).) Thus ϕ will be constant and so from (10) the angular velocity will be constant about both x and y. Since we are assuming that only one fore-aft jet is working it is now necessary to bring all the angular velocity along the axis perpendicular to this fore-aft jet and in the proper sense so that torque about this one axis will remove all the angular velocity. To move the angular velocity to the proper direction we use (12) which says that for every degree of roll angle the ϕ direction (which locates ω_{xy} with respect to the vehicle hence torque axes), will change by $-(1 - \frac{C}{A})$ degrees and stay there. Thus if the rotation ended up at the end of Step 1 along the +y direction and we wanted it along the -y direction we would want to change ϕ by 180° and with $C/A = 1/6$ this would require a roll of $\frac{6}{5}$ $180 = 216^\circ$ plus or minus any multiple of 360°. Therefore, we have:

Step 2 Roll the vehicle through the proper angle to place the angular velocity along the direction about which the single fore-aft jet can remove angular rates.

At the conclusion of Step 2, the vehicle will be rotating about the xy plane axis where the fore-aft jet can remove rates so:

Step 3 Apply the fore-aft thrust until the xy plane angular velocity goes to zero.

The angular velocity of the vehicle is now zero so we are now ready to point the $\bar{z}$ axis at the commanded star. With only one fore-aft jet, the procedure is to first roll the spacecraft about $\bar{z}$ until the star lies in the $\bar{z}$ / fore-aft jet plane, then the roll stopped. Again, since the vehicle will roll past the desired point each revolution, there is plenty of time to start the slow roll with a single jet and then rotate the jet 180° to stop the roll. Therefore, we have

Step 4   Roll the vehicle through the angle necessary to place the star in the plane of $\bar{z}$ and the operating fore-aft jet.

Now all that remains to do is to apply a fore-aft torque to slowly pitch the $\bar{z}$ axis into the desired direction and then stop this pitch rate with a reverse torque in this plane. Again, a long delay is tolerable to reverse the jet direction since the vehicle will pass the desired direction once each revolution.

The $\bar{z}$ axis is now pointed in the correct direction with the vehicle at zero angular rate. Therefore, the final step is then to apply a long roll torque to build up spin speed for later attitude stabilization during the thrust application.

Step 5   Apply roll torque to build up desired spin speed for attitude stabilization during thrusting period.

If the boost is to be applied with the 20,000 lb. main engine, a 1 foot c. g. uncertainty gives a 20,000 lb-ft torque and Note No. 38 indicates that a spin speed of about 3 rad/sec will be necessary. On the other hand if the boost were applied by a single fore-aft jet the moment would only be 100 x 7 = 700 lb-ft which is a factor of 30 less. However, the angle varies as the square of spin speed (equation (57), Note 38), so the minimum spin with only one reaction jet would be about 0.6 rad/sec and with 4 would be about 1.2 rad/sec.

APOLLO NOTE NO. 63

G. F. Floyd
10 May 1963

DSIF CAPABILITY ON TRANS-EARTH TRAJECTORY

As shown in Note No. 26, the sensitive directions for re-entry dive angle and miss along (range miss) are so close while in the earth's sphere of influence that independent control of the two orbit parameters is impractical. Luckily, however, the results of Note No. 62 show that this high correlation also exists back at lunar injection. At lunar injection we have for the coefficients along the most sensitive direction,

$$\frac{\Delta M_{||}}{\Delta \delta_f} = 220 \text{ km per degree} \tag{1}$$

while after we enter the earth sphere of influence, we have, from page 4 of Note No. 26:

$$\frac{\Delta M_{||}}{\Delta \delta_f} = 210 \text{ km/degree} \tag{2}$$

and the sensitivity ratios are of the same sign. Therefore, the ratios are so nearly the same that midcourse corrections to correct the effect of injection errors or re-entry angle will automatically correct the range miss. Further justification of this conclusion is furnished by the fact that from Note No. 53 we see that the range dispersion with a zero-lift re-entry will be several hundred miles at re-entry angle dispersion of 0.1°. Therefore, there is no gain in holding the re-entry miss much smaller than this value. This result is, of course, fortunate since we can't control the two separately anyhow.

Also as shown in Note No. 26, the sensitivity of the out of plane miss (track miss) is very much lower than that of the re-entry angle. Therefore, if we can measure and control the re-entry angle to 0.1° we will be able to control the track miss to better than a fraction of a nautical mile. Consequently, the following analysis of the DSIF capability on the return trajectory will be devoted entirely to the variance in the re-entry dive angle.

The basic formulas for the variance in the estimated value of (δ_f) from $\dot{n}$ data observed over the range interval R_2 to R_1 are derived in Note No. 5 and the results of a computer evaluation of these formulas is given as $\phi(R_2, R_1)$ of Figure 1 and Figure 2. The variance in (δ_f) related to the plotted function $\phi(R_1, R_2)$ is,

$$\sigma^2(\delta_f) = \frac{\sigma^2(\dot{R})(R_2 - R_1)\,\phi(R_2, R_1)}{N}$$

where

$$N \triangleq \text{number of samples} = \frac{T(R_2) - T(R_1)}{t_{correlation}}$$

and T(R) is the time-to-go at range R, which is plotted in Figure 3.

The optimum method of combining old and new measurements when there is an intervening imperfectly executed change is derived in Note No. 57, and this method was used in calculating the performance during a typical midcourse correction system for earth return.

In order to determine the limitations of the DSIF, the assumption was made that the CM was under completely manual control with no autopilot and no integrating accelerometers. The execution of the ground derived commands is then accomplished by pointing the spacecraft at the ordered star, spinning it to get spin stabilization (see Note No. 38), and applying the boosts for a commanded time interval. With such crude control, the major error that results would be that of the assumed 10% engine thrust level uncertainty since the errors due to poor directional control arising from unbalanced torques (c. g. shifts or locked over nozzles), were shown in Note 38 to be less than 12° and with the boost ordered in the most sensitive direction (Note 22), these directional errors would result in percentage error of less than 2% which is negligible compared with the 10% magnitude error that would result without a longitudinal integrating accelerometer.

The assumption of very poor execution of the commands puts a real premium on navigation accuracy and short smoothing time in order to keep the midcourse correction fuel requirements within the fuel pad limits. The reason for this is shown in Figure 4 which is a plot of the

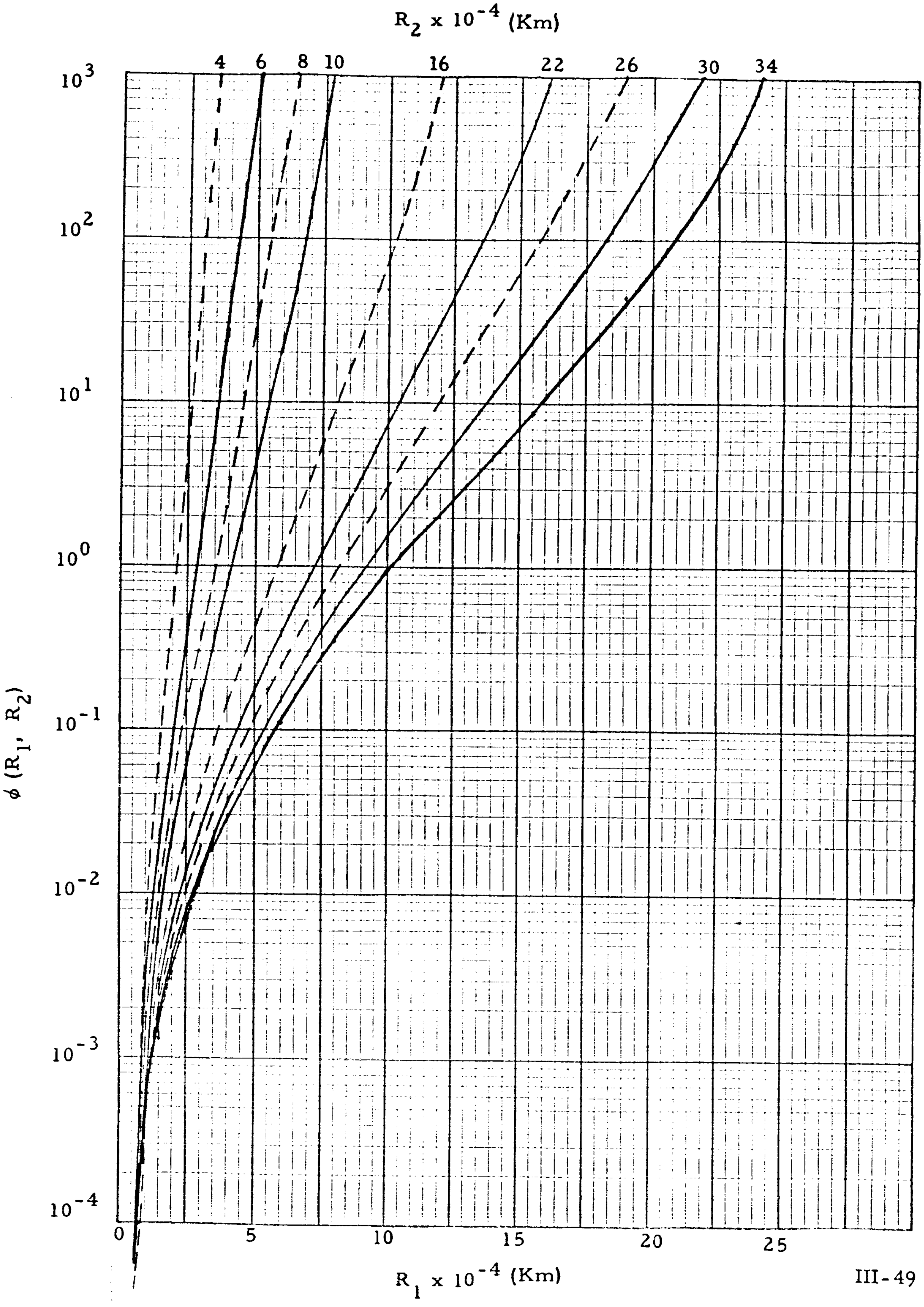
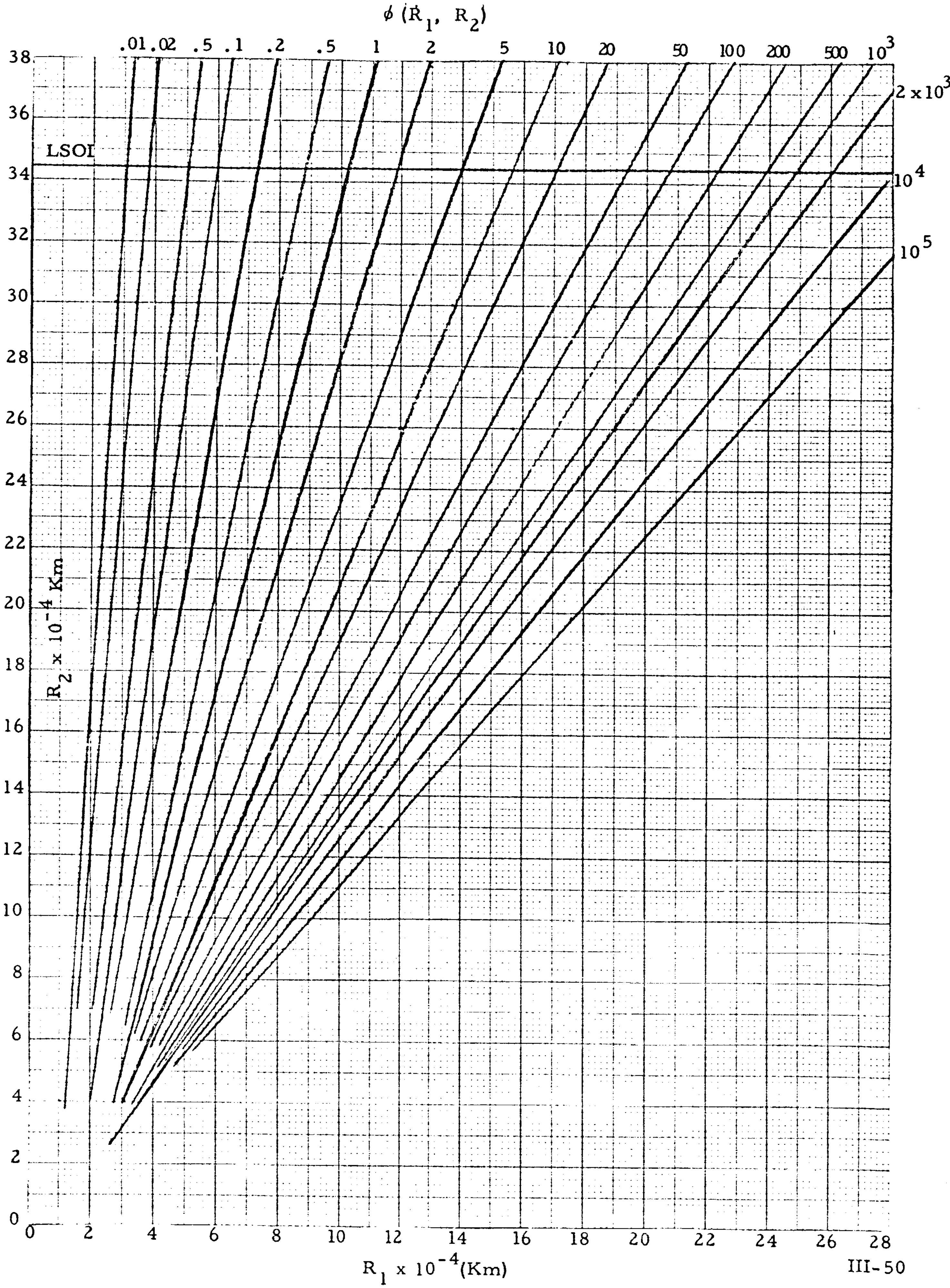
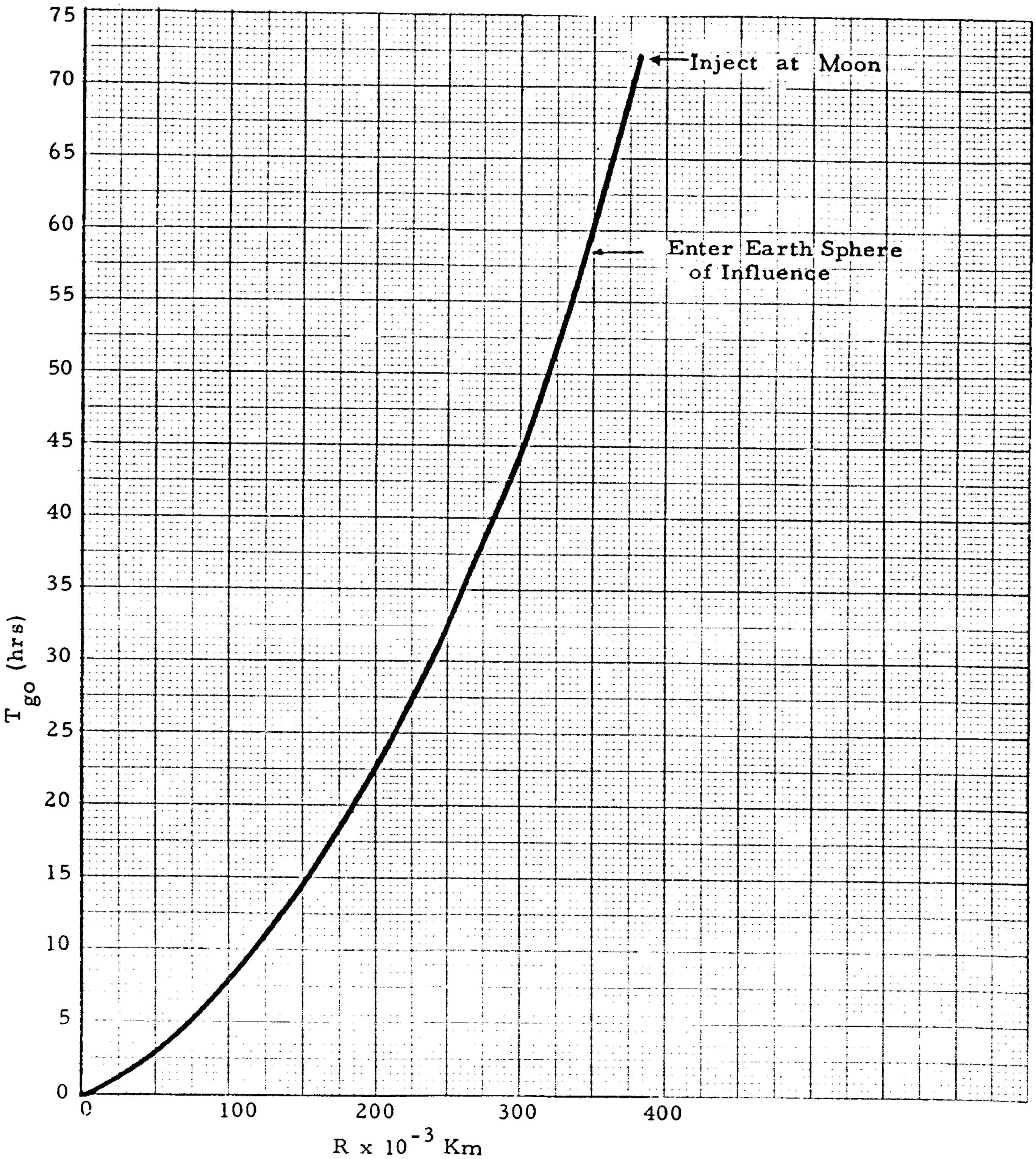

$R_2 \times 10^{-4}$ (Km)
4
6
8
10
16
22
26
30
34
10^3
10^2
10^1
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
$\phi(R_1, R_2)$
0
5
10
15
20
25
$R_1 \times 10^{-4}$ (Km)

Figure 1.



$\phi(R_1, R_2)$
.01 .02 .5 .1 .2 .5 1 2 5 10 20 50 100 200 500 10^3
2×10^3
10^4
10^5
LSOI
$R_2 \times 10^{-4}$ Km
0 2 4 6 8 10 12 14 16 18 20 22 24 26 28 30 32 34 36 38
0 2 4 6 8 10 12 14 16 18 20 22 24 26 28
$R_1 \times 10^{-4}$(Km)

Figure 2.



75
70
65
60
55
50
45
40
35
30
25
20
15
10
5
0
T_{go} (hrs)
Inject at Moon
Enter Earth Sphere of Influence
0
100
200
300
400
$R \times 10^{-3}$ Km

Figure 3.

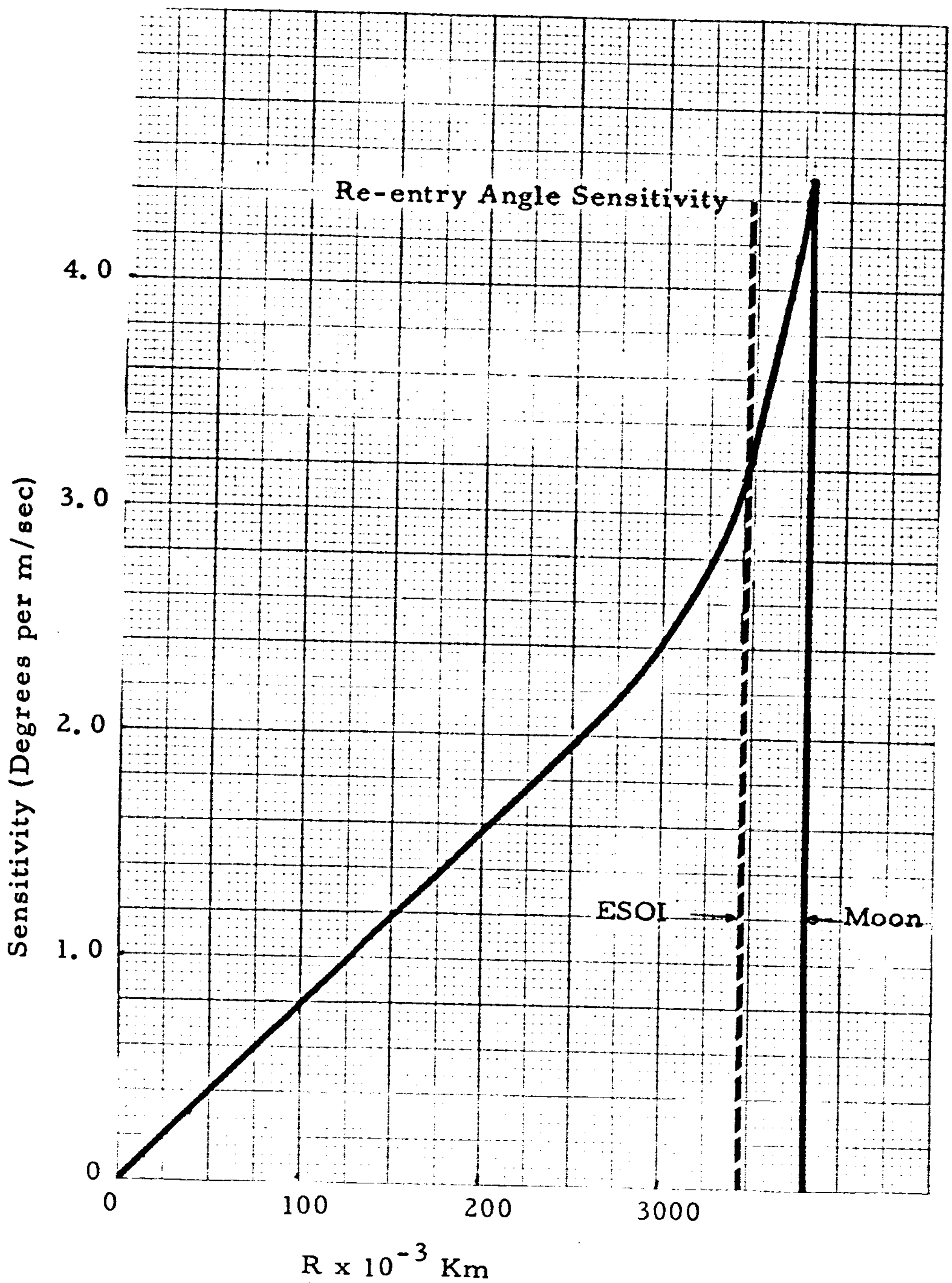

Re-entry Angle Sensitivity
4.0
3.0
2.0
1.0
0
Sensitivity (Degrees per m/sec)
ESOI
Moon
0
100
200
3000
R x 10^{-3} Km

Figure 4.

re-entry angle sensitivity coefficient in degrees per meter/sec as a function of range from the earth. At injection the coefficient is 4.5°/m/sec and it very rapidly falls to 3.0°/m/sec as the spacecraft goes through the lunar sphere of influence. Thus even if the injection error could be completely corrected when the vehicle reached the earth sphere of influence, the correction required would be 1.5 times the injection error and with a 10% autopilot the injection error will be $\frac{1}{10} \times 3000 = 300$ ft/sec so the midcourse ΔV required will be at least 450 ft/sec.

No attempt has been made yet to optimize the midcourse correction schedule other than just intuitive guesses at one which will give satisfactory performance. The results with such a "guessed at" program are shown in Table 1.

Table 1.

Time Till Re-entry (hrs)	DSIF 1σ Uncertainty in Re-entry Angle	Correction No.	Commanded Boost ft/sec	Resultant Error in Re-entry Angle (degrees)
72	DNA	Injection	3000	450
50	1.1	1	550	45
36	1.1	2	69	4.6
24	.7	3	10	.8
16	.3	4	2.5	.3
8	.09	5	1.5	.09
2	.02	-	-	-

As the table shows, the capabilities of the DSIF are so good that even with very poor execution of commands, resulting in very large initial errors, large errors in the execution of each command (so that old smoothing data is not useful, see Note 57), the DSIF still can get the spacecraft within the 0.1° tolerance at eight hours to go (spacecraft at 15 earth radii).

It will be noted that five corrections are required. This is not due to DSIF uncertainties but simply because of the large execution error that was assumed. With a 10% execution error, and an initial error of 450^{o}, it would take four perfectly computed corrections to bring the result to .04 whereas with present DSIF accuracy, tracking errors stretch this to only 5. The total midcourse boost needed is 633 ft or about twice the injection error as compared with 450 ft/sec if there were DSIF errors.

In summary, it appears that with DSIF orbit prediction accuracy, it should be possible to make a safe zero lift re-entry with manual execution of the commanded maneuvers. Also it should be emphasized that if there were a single longitudinal integrating accelerometer, the execution error would be reduced by a factor of 5 to the 2% associated with spin stabilization. In this case the re-entry angle error reduction of Table 1 would look much better and reach the ; 1^{o} goal at more than 12 hours out.